Non-Thermal Dark Matter and Effective Coupling to Nucleons

1. **Gravitino thermalization**

   a) Gravitinos can thermalize if the scattering of SM particles and their superpartners is vigorous enough. Assume that $T \gtrsim m_{soft}$ and that both the SM and its superpartners are in thermodynamic equilibrium. Estimate the temperature above which thermalization occurs, assuming it is larger than $m_{3/2}$.

   b) Do the same, but assuming that it is less than $m_{3/2}$.

   c) Which of these results is the relevant one for $m_{3/2} < M_{\rm Pl}$?

2. **Non-thermal DM (instantaneous approximation)**

   a) Suppose we have the non-thermal scenario described in Section 2 of `notes-4`, where a massive particle P dominates the energy density of the Universe until is decays at time $t = \tau_P = 1/\Gamma_P$. A good approximation to the evolution equations describing this process can be obtained by treating the decay as being instantaneous, transferring all the energy in P particles to radiation. Match the energy densities just before and after the decay to find the approximate reheating temperature.

   b) What is the number density of P particles just before it decays?

   c) The interpretation of Eq. (13) in `notes-4` is that each P decays produces an average number of $\epsilon \lll 1$ DM particles. (*e.g.* $P \to \chi\chi$ happens once per thousand decays implies $\epsilon = 1/500$.) Use this to estimate the number density of DM particles just after the P density decays.

   d) If $T_{RH} \ll T_{fo} < m_\chi$ for χ, the DM particles created by the P decays will not annihilate any further. Use this to estimate the yield of DM particles today.

3. **QCD and Nucleons**
The fundamental fields of QCD are quarks and gluons, but we know that these become strongly-coupled at energies $E \sim \Lambda_{QCD} \simeq 1$ GeV. At energies below this, the relevant degrees of freedom are nucleons and mesons. We can describe their low-energy interactions perturbatively by writing an effective field theory for them. The tricky part is that we don't know how to predict what these interactions should be from the underlying theory of quarks and gluons because of their very strong interactions. Even so, we can match many aspects between the two theories by making use of the underlying symmetry properties. The remaining gaps can be filled by lattice simulations of QCD.

   a) Let's apply these ideas to direct detection. Suppose the underlying DM-quark interactions are

$$-\mathscr{L} \supset \bar{\chi}\chi \left(\sum_{q=u,d,s} d_q\, \bar{q}q + \sum_{Q=c,b,t} d_Q\, \bar{Q}Q \right) . \tag{1}$$

   We want to convert this into an effective χ-nucleon interaction. For the light quark couplings, lattice QCD gives

$$\langle \tilde{n}(p_4)|\, m_q \bar{q}q \,|\tilde{n}(p_2)\rangle = m_{\tilde{n}} f_{T_q}^{(\tilde{n})}\, \bar{u}_4 u_2 \tag{2}$$

   for some constants $f_{T_q}^{(\tilde{n})}$, and $\bar{u}_4 = \bar{u}(p_4, s_4)$ and $u_2 = u(p_2, s_2)$ are polarization spinors. We also know that $\langle \tilde{n}(p_4)|\, \bar{\tilde{n}}\tilde{n} \,|\tilde{n}(p_2)\rangle = \bar{u}_4 u_2$ for nucleon fields. In the case of $d_Q = 0$ for all the heavy quarks, what χ-nucleon effective interaction would reproduce the effects of the underlying quark interactions (*i.e.* give the same matrix elements between nucleon states) given Eq. (2)?

   b) In general, there will also be a contribution to the effective χ-nucleon interaction from heavy quarks. The effect of integrating them out at one-loop order is to generate the effective interactions obtained by making the replacements

$$\bar{Q}Q \to -\frac{2\alpha_s}{24\pi\, m_Q}\, G^a_{\mu\nu} G^{a\,\mu\nu} \tag{3}$$

   wherever $\bar{Q}Q$ appears in the Lagrangian. What is the resulting low-energy effective Lagrangian in terms of only the light quarks q and the gluon field $G^a_{\mu\nu}$?

   c) Given that we also have an induced $\bar{\chi}\chi G^a_{\mu\nu} G^{a\,\mu\nu}$ interaction, we need to relate it as well to a nucleon interaction. Let us define

$$\langle \tilde{n}|\, G^a_{\mu\nu} G^{a\,\mu\nu} \,|\tilde{n}\rangle = -\frac{8\pi}{9\alpha_s}\, f_{T_G}^{(\tilde{n})} . \tag{4}$$

   It turns out that we can relate $f_{T_G}^{(\tilde{n})}$ to the constants for light quarks. To do this, we should look at the divergence of the *dilatation current*, which is just the trace of the improved energy-momentum tensor, $\Theta^{\mu}{}_{\mu}$ (see Ch.19.5 of Peskin&Schroeder). The Lagrangian of QCD is invariant under scale transformations ($x \to \lambda x$, $q \to \lambda^{-3/2} q$, $A^a_\mu \to \lambda^{-1} A^a_\mu$) when the quark masses vanish. Quantum effects also break the invariance in the form of the running of strong coupling. As a result, the divergence of the dilatation current is equal to these breaking effects:

$$\Theta^{\mu}{}_{\mu} = \sum_{q=u,d,s} m_q \bar{q} q - \frac{9\alpha_s}{8\pi} G^a_{\mu\nu} G^{a\,\mu\nu} \ . \tag{5}$$

The first term comes from the explicit breaking by quark masses while the second term comes from the quantum breaking effect, with the coefficient of the second piece being proportional to the beta function of QCD with three light flavours. We can also write a dilatation current for the low-energy theory with nucleon fields:

$$\Theta^{\mu}{}_{\mu} = m_p \bar{p} p + m_n \bar{n} n + \dots \ . \tag{6}$$

Since both descriptions of QCD describe the same underlying theory, the dilatation current operators in the two cases must be equivalent in the sense that they have the same matrix elements. Use this fact to relate $f^{(\tilde{n})}_{T_G}$ to the $f^{(\tilde{n})}_{T_q}$ constants.

d) Put all these pieces together to find the effective χ couplings to protons and neutrons.