The Jeans Instability

1 Gravitational Collapse of Hot Gas

Hot gases over a certain size are likely to undergo gravitational collapse. Consider a gas at a certain temperature. The thermal fluctuations of the gas is mediated by sound waves propagating at speed c_s. Density fluctuations of the gas will tend to induce gravitational collapse but these will tend to be smoothed out by thermal fluctuations. However, if the density fluctuation is on a length scale that is so large that sound waves cannot prevent parts of the gas from free fall then the gas will undergo gravitational collapse. This is the essence of the Jeans instability. In what follows, we shall flesh this out in more detail.

The Equations of Fluid Dynamics

Consider a fluid with density ρ, pressure P and velocity $\mathbf{v}$ in a gravitational potential ϕ. Then the equation of motion of such a fluid is given by Euler's equation (which is just Newton's equation for the fluid):

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla \phi. \tag{1}$$

In addition, we also have the equation of continuity and Poisson's equation for gravity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

$$\nabla^2 \phi = 4\pi G \rho, \tag{3}$$

where G is Newton's constant.

Linearized Equations

Consider a uniform and static ideal gas. Then consider the following perturbations

$$\begin{aligned}\rho &= \rho_0 + \rho_1 \\ P &= P_0 + P_1 \\ \mathbf{v} &= \mathbf{v}_1 \\ \phi &= \phi_0 + \phi_1\end{aligned}$$

where the background field values have been denotes by the subscript 0 (unless it vanishes) and the small perturbations have been labelled by the subscript 1.

If one counts the number of independent variables and the the number of equations above, it is clear that this system of equations is underdetermined. So, it needs to be supplemented by the equation of state

$$P = P(\rho, S)$$

where S is the entropy. If we assume an ideal gas and the perturbations are adiabatic then the density and pressure perturbations are related by,

$$P_1 = c_s^2 \rho_1,$$

where $c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S,0}$ is the square of the speed of sound in the gas.

- What are the linear versions of equations (1), (2) and (3)?
- Combine these equations to obtain a second order wave equation (with source) for the density fluctuation.[1]
- Assume a plane wave ansatz for the density fluctuation. Show that the dispersion relation that follows from the wave-equation is

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0.$$

- Derive the Jeans length λ_J, i.e. the minimal length above which fluctuations become unstable.
- The Jeans mass is the mass of the matter that is contained in a sphere of radius $\frac{1}{2}\lambda_J$. Derive the Jeans mass.

2 Decoupled Species

a. Suppose a massive particle species χ has number density $n_\chi(t_{fo})$ at time t_{fo} when the temperature of the Universe is $T_{fo} < m_\chi$.

[1] In deriving the wave equation ignore the zeroth order result of Poisson's equation: $\nabla^2\phi_0 = 4\pi G\rho_0$. The uniformity of ϕ_0 would imply that we set $\rho_0 = 0$ invalidating the whole exercise. This is knows as the "Jeans swindle". In a real hot gas there would be other forces present which ensures the consistency to this analysis.

i) If the particle species doesn't interact at all with the cosmological plasma after t_{fo}, what is the number density of that species today in terms of the scale factor $a(t)$ then and now? What is its contribution to the energy density today?

ii) The entropy per comoving volume sa^3 (where s is the entropy density) remains constant through much of the evolution of the Universe. Assuming it is constant after t_{fo}, rewrite the result from part i) in terms of the temperature and number of entropic degrees of freedom g_{*s} at t_{fo} and those today (at time t_0).
Hint: $s = 2\pi^2 g_{*s} T^3/45$.

iii) If we had $T_0 \ll m_\chi < T_{fo}$, how would your previous answers differ?

b. Suppose a massless relativistic particle species V in equilibrium with the cosmological plasma at time t_{fo} suddenly stops interacting at all with itself or anything else. After this happens, the expansion of spacetime will dilute the number density of V particles as a^{-3} energy of each V particle will redshift as a^{-1}.

i) What is the contribution of V to the energy density of the Universe today?

ii) Rewrite this result in terms of temperatures and g_{*s} factors assuming entropy conservation after t_{fo}.

iii) One can show that the distribution function for V particles tracks the equilibrium form, but with a different effective temperature for the V particles. What is this effective temperature today in terms of the photon temperature and the number of effective degrees of freedom g_*?
Hint: $\rho(T) = \pi^2 g_*(T)\, T^4/30$ *for this distribution function.*