# Lecture Note #2: Thermal Dark Matter Creation

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The existence of a new stable, neutral, massive particle is not enough to explain the observed dark matter. There must also be a mechanism through which the proper density of this particle was created in the early Universe. It turns out that the expansionary cooling of the Universe can achieve this provided the DM particle has the right mass and interactions. The corresponding mechanism is called *thermal freeze out*, and we will describe it in detail here.

The assumption underlying thermal freeze out is that the DM particle, (which we will refer to as $\chi$) was once in thermodynamic equilibrium with the hot plasma of SM particles created after inflation. During this period, the Universe was radiation-dominated with a temperature so large that the DM particle was also highly relativistic. As the Universe cooled below the mass of $\chi$, the annihilation reactions $\chi\chi \leftrightarrow SM\,SM'$ could no longer keep up with the expansion of the Universe and effectively turned off, leaving a *relic density* of $\chi$ particles much larger than their equilibrium value. In this case, the $\chi$ particle is said to have *frozen out* of the thermal bath, and the leftover density makes up the DM.

# 1 Equilibrium and Departures from It

To understand how thermal freeze out works, let us first go over what thermodynamic equilibrium means in the early Universe and how to compute deviations from it.

## 1.1 Equilibrium

All evidence seems to indicate that the Universe was a hot soup of elementary particles at some point early in its history. The most useful way to describe this soup is statistical mechanics, where the average properties of the $i$-th particle species are completely described by the *distribution function* $f_i(t, \vec{x}, \vec{p})$. For example, the local number density, energy density, and partial pressure of that species are

$$n_i(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, g_i\, f_i(t, \vec{x}, \vec{p}) \tag{1}$$

$$\rho_i(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, g_i\, E_i(\vec{p})\, f_i(t, \vec{x}, \vec{p}) \tag{2}$$

$$p_i(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, g_i\, \frac{\vec{p}^2}{3E_i(\vec{p})}\, f_i(t, \vec{x}, \vec{p}) \tag{3}$$

$$\tag{4}$$

where $g_i$ is the number of internal degrees of freedom; spins, colours, and such.

You have already encountered distribution functions in thermodynamics. In the Grand Canonical Ensemble (GCE), the distribution function for a particle species $i$ is

$$f(t, \vec{x}, \vec{p}) = f_{GCE}(E) = \left[e^{(E-\mu_i)/T_i} \mp 1\right]^{-1} , \tag{5}$$

where $T_i$ is the temperature, $\mu_i$ is the chemical potential, and the minus (plus) sign corresponds to when the species is a boson (fermion). The temperature fixes the mean energy density of the particle species, and the chemical potential fixes the mean number density.

Equilibrium is the state the system would attain if we were to leave it to settle for an infinitely long time, possibly subject to constraints. The GCE distribution corresponds to the state of equilibrium attained by a system of particles allowed to interact very weakly with a large reservoir, exchanging both particles and energy with it, while holding fixed the total energy and number of particles (in the system plus reservoir). This is precisely the situation of the cosmological plasma; the system is the small local region we are studying, and the bath is the rest of the Universe (or at least the rest of our Hubble volume).

Given this definition, you may be wondering whether it makes sense to apply the tools of thermodynamic equilibrium to the early Universe. After all, the Universe has a finite age and is expanding, with its contents cooling along the way. In many cases, the answer is yes to an excellent approximation. The condition for this approximation to hold for a particle species $i$ is that the interaction rates driving the distribution function $f_i$ towards the equilibrium value must be much faster than the rate of spacetime expansion, characterized by the Hubble rate $H$. [1] In this case, the effect of the expansion can be treated as an adiabatic change in the values of $T_i$ and $\mu_i$ appearing in the equilibrium distribution of Eq. (5).

When many particle species interact with each other through rapid elastic scattering, they reach a kinetic equilibrium with all species having the same temperature. Reactions that change particle number maintain chemical equilibrium. In particular, if the reaction $(a + b + \ldots \leftrightarrow c + d + \ldots)$ is fast, the chemical potentials of those species are related by

$$\mu_a + \mu_b + \ldots = \mu_c + \mu_d + \ldots \quad . \tag{6}$$

Full thermodynamic equilibrium requires both kinetic and chemical equilibrium.

We define the temperature of the cosmological plasma to be the temperature of the photons, $T = T_\gamma$. These are observed to have a thermal distribution given by Eq. (5) with $\mu_\gamma = 0$. At high temperatures, the rest of the plasma can usually be treated as an unpolarized gas of (feebly-interacting) elementary particles. Thus, the state of any single particle is completely characterized by its 3-momentum $\vec{p}$, with $E = \sqrt{m_i^2 + \vec{p}^2}$. A given particle species is said to be in thermodynamic equilibrium with the cosmological plasma if its distribution function $f_i(t; \vec{x}, \vec{p}) = f(t; E)$ is given by the GCE distribution of Eq. (5) with $T_i = T$ and some $\mu_i$. For a particle species $\psi$ that annihilates quickly with its anti-particle $\bar{\psi}$ to photons, chemical this implies that $\mu_{\bar{\psi}} = -\mu_\psi$. Thus, a non-zero chemical potential corresponds to an asymmetry in the densities of particles and antiparticles. When a particle is its own antiparticle (in chemical equilibrium with photons), this also implies $\mu_\psi = 0$.

[1] During radiation and matter domination, $H^{-1}$ corresponds to the age of the Universe.

Now that we know the microstates of the $i$-th particle species and its distribution function in thermodynamic equilibrium, we can compute the equilibrium number density. Taking $\mu_i = 0$ for now and inserting the equilibrium distribution into Eq. (1), one finds

$$n_{i_{eq}} = \begin{cases} \left\{ \begin{smallmatrix} 1 \\ 3/4 \end{smallmatrix} \right\} g_i \frac{\zeta(3)}{\pi^2} T^3 & ; \quad T \gg m_i \\ g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T} & ; \quad T \ll m_i \end{cases} , \tag{7}$$

where the 1 (3/4) is for bosons (fermions) and $\zeta(3) \simeq 1.202$. At high temperatures we see that $n_i \sim T^3$, while at low temperatures we have $n_i \propto e^{-m_i/T}$ corresponding to the usual Boltzmann suppression of states with $E \simeq m_i \gg T$.

## 1.2 Departures from Equilibrium

We turn next to departures from thermodynamic equilibrium in the early Universe, which can occur when the equilibrating reaction rates become slower than the Hubble expansion. Such departures are described by deviations in the distribution functions from the equilibrium GCE values. The technology that we will develop to do so will allow us to compute the thermal freeze out of DM.

Full thermodynamic equilibrium implies both *chemical equilibrium* and *kinetic equilibrium*. Chemical equilibrium means that the relative number densities of the species in the system match the equilibrium values. Kinetic equilibrium means that the distribution has the same energy dependence as the equilibrium value, or equivalently

$$f_i(t, E) = \xi_i(t) f_{i_{eq}}(E) , \tag{8}$$

where $\xi(t)$ is some function of time alone (but not energy), and the equilibrium distribution depends on time only through the slow variation of $T$ and $\mu_i$. For thermal freeze out of DM, we will see that chemical equilibrium is typically lost before kinetic equilibrium.

Departure from thermodynamic equilibrium in the distribution $f_i$ is described by a Boltzmann equation of the form

$$L[f_i] = C[f_i; \{f_j\}] . \tag{9}$$

The left side is called the Liouville term and the right side is called the collision term. For a non-relativistic system, the Liouville term is given by

$$\left( \frac{\partial}{\partial t} + \frac{dx_k}{dt} \frac{\partial}{\partial x_k} + \frac{dp_k}{dt} \frac{\partial}{\partial p_k} \right) f_i . \tag{10}$$

We see that it is just a total time derivative. The relativistic generalization is

$$L[f_i] = \left( p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \right) f_i , \tag{11}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol for the background spacetime. Unlike the Liouville term, the collision term $C[f_i; \{f_j\}]$ is process-dependent and is a function of the distributions of other particles present in the plasma. It describes scattering and decay processes that modify the distribution $f_i$. Putting these pieces together, we see that the Boltzmann equation is just a statement that the time variation of the distribution function is the result of particle interactions.

Starting from the Boltzmann equation, we can integrate both sides over $d^3p_\chi$ to get an equation for the time variation of the number density $n_\chi$ of the particle species $\chi$ in the early Universe. The result is

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \widetilde{C}[f_\chi; \{f_j\}] \ . \tag{12}$$

The $3Hn_\chi$ term on the left describes the dilution of the $\chi$ density by the expansion of spacetime; if $\widetilde{C} = 0$ we would have $n_\chi \propto a^{-3}$, where $a(t)$ is the expansion factor. The collision term on the right has a contribution from every process in the plasma that changes the number of $\chi$ particles. For the process $\chi + a + \ldots + b \leftrightarrow i + \ldots + j$, the contribution to the collision term is given by

$$\begin{aligned} \Delta\widetilde{C} \ &= \ -\int (d\Pi_\chi d\Pi_a \ldots d\Pi_b)(d\Pi_i \ldots d\Pi_j)(2\pi)^4\delta^{(4)}(p_\chi + p_a + \ldots + p_b - p_i + \ldots + p_j) \\ &\times \frac{1}{S}\Big[|\widetilde{\mathcal{M}}_{\chi+\ldots+b\to i+\ldots+j}|^2 f_\chi f_a \ldots f_b(1 \pm f_i)\ldots(1 \pm f_j) \\ &\quad - |\widetilde{\mathcal{M}}_{i+\ldots+j\to\chi+\ldots+b}|^2 f_i \ldots f_j(1 \pm f_\chi)\ldots(1 \pm f_b)\Big] \ , \end{aligned} \tag{13}$$

where

$$d\Pi_i = g_i\, d^3p_i/(2\pi)^3\, 2E_i \tag{14}$$

is the Lorentz-invariant phase space measure, and $|\widetilde{\mathcal{M}}_{I\to F}|^2$ is the squared matrix element for the reaction $I \to F$ *averaged over all initial and final internal degrees of freedom*. We also see that the forward reaction $\chi + \ldots + b \to i + \ldots + j$ is weighted by the phase space distributions of all the particles in the initial state and factors of $(1 \pm f_i)$ for each particle in the final state. The sign here is positive if species $i$ is a boson and negative if it is a fermion. These factors account for Pauli blocking (reduced final-state phase space for fermions due to Pauli exclusion) or stimulated emission (enhanced phase space by Bose condensation). The symmetry factor $S$ accounts for identical particles, and picks up a factor of $N!$ for every set of $N$ identical species in the initial or final state.

The collision term of Eq. (13) is very complicated, but it can be simplified greatly by making a few reasonable approximations. In most cases of interest we have $f_i \ll 1$, meaning that $(1 \pm f_i) \simeq 1$. It is also usually the case that the effects of CP violation are numerically small. If so, we have $|\widetilde{\mathcal{M}}_{\chi+\ldots+b\to i+\ldots+j}|^2 \simeq |\widetilde{\mathcal{M}}_{i+\ldots+j\to\chi+\ldots+b}|^2 := |\widetilde{\mathcal{M}}|^2$. Plugging this back into Eq. (13), we get

$$\begin{aligned} \Delta\widetilde{C} \ &= \ -\int (d\Pi_\chi d\Pi_a \ldots d\Pi_b)(d\Pi_i \ldots d\Pi_j)(2\pi)^4\delta^{(4)}(p_\chi + p_a + \ldots + p_b - p_i + \ldots + p_j) \\ &\quad \times \frac{1}{S}|\widetilde{\mathcal{M}}|^2(f_\chi \ldots f_b - f_i \ldots f_j) \ . \end{aligned} \tag{15}$$

This is starting to look like the total cross sections for the forward and reverse reactions, integrated over the initial-state phase space and weighted by the initial-state distribution functions. To see this, compare Eq. (15) to the expression for scattering cross sections given in `notes-0`.

## 2 Thermal Freeze Out of Dark Matter

We now have all the tools needed to study the thermal freeze out of dark matter. To be concrete, we will assume that the DM particle $\chi$ is its own antiparticle (so that $\bar{\chi} = \chi$) and that the only relevant $\chi$-number-changing reaction is $\chi\chi \leftrightarrow f\bar{f}$ for some SM fermion $f$. The relevant collision term is therefore

$$\Delta\widetilde{C} = -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})(d\Pi_f d\Pi_{\bar{f}})(2\pi)^4\delta^{(4)}(\ldots)\,|\widetilde{\mathcal{M}}|^2 \left(\frac{1}{2}\times 2\right)(f_{\chi_1}f_{\chi_2} - f_f f_{\bar{f}})\;. \qquad (16)$$

The factor of $(2\times 1/2) = 1$ accounts for the fact that the number of $\chi$ changes by two units in this reaction, but is cancelled by the symmetry factor for the two identical particles in the initial state.

We are specifically interested in the behaviour of the collision term when $m_f \ll T \lesssim m_\chi$, assuming that freeze out only occurs when the particle is non-relativistic. In this case, we can reliably approximate the equilibrium distributions of $f$ and $\chi$ by simple Boltzmann factors ($1/(e^{-m/T} \mp 1) \simeq e^{-m/T}$), and we take the SM fermions $f$ and $\bar{f}$ to be in thermodynamic equilibrium. Together with energy conservation in the reaction, this gives

$$f_f f_{\bar{f}} \simeq e^{-(E_f+E_{\bar{f}})/T} = e^{-(E_1+E_2)/T} \simeq f_{1_{eq}} f_{2_{eq}}\;, \qquad (17)$$

where $f_{1_{eq}}$ and $f_{2_{eq}}$ are the (Boltzmann) distribution functions that $\chi_1$ and $\chi_2$ would have if they were in thermodynamic equilibrium. Applying this to Eq. (16), we get

$$\begin{aligned}\Delta\widetilde{C} &= -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})\,(f_{\chi_1}f_{\chi_2} - f_{1_{eq}}f_{2_{eq}})\left[\int (d\Pi_f d\Pi_{\bar{f}})(2\pi)^4\delta^{(4)}(\ldots)|\widetilde{\mathcal{M}}|^2\right] \\ &= -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})\,(f_{\chi_1}f_{\chi_2} - f_{1_{eq}}f_{2_{eq}})\,(\sigma v)\,(2E_1 2E_2) \qquad (18) \\ &= -\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\,g_\chi^2\,(f_{\chi_1}f_{\chi_2} - f_{1_{eq}}f_{2_{eq}})\,(\sigma v)\;.\end{aligned}$$

In this expression, we have made use of the standard definition of the cross section [2], described in `notes-0`. Recall that $v$ corresponds to the relative velocity of the incoming particles, and is given by $v = \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}/E_1E_2$.

We can simplify this last expression even further by making the assumption that the DM particles remain in kinetic equilibrium throughout the freeze out process. Using Eq. (8) we get

$$\Delta\widetilde{\mathcal{C}} = -(n_\chi^2 - n_{\chi_{eq}}^2)\langle\sigma v\rangle\;, \qquad (19)$$

where the *thermal average* of an operator $\mathcal{O}(p_1, p_2)$ is defined to be

$$\langle \mathcal{O} \rangle = \frac{1}{n_{eq}^2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_1}{(2\pi)^3} \, g_1 g_2 \, f_{1_{eq}} f_{2_{eq}} \, \mathcal{O}(p_1, p_2) \; . \tag{20}$$

Note that it is defined with respect to the equilibrium distributions. Putting everything together, the final form of the Boltzmann equation for the cosmological evolution of the $\chi$ density is

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle (n_\chi^2 - n_{\chi_{eq}}^2) \; . \tag{21}$$

This will be our key equation for thermal freeze out.

Before trying to solve Eq. (21), it is worth thinking a little about the solution when the cross section is large enough that $\chi$ remains in equilibrium until $T$ falls below $m_\chi$.. At temperatures well above the DM mass, $T > m_\chi$, the equilibrium number density $n_{\chi_{eq}} \sim T^3$ is unsuppressed and the collision term on the right will be much larger than the Hubble term. In this case, the number density of $\chi$ will be driven to its equilibrium value; $n_\chi \simeq n_{\chi_{eq}}$. On the other hand, as the temperature cools below the mass, $T \lesssim m_\chi$, the equilibrium density of $\chi$ becomes exponentially suppressed and the collision term becomes less important. The number density will initially track the equilibrium value until the Hubble term in Eq. (21) becomes larger, at which point the annihilation process effectively turns off. Physically, the number density of $\chi$ becomes so small that the mean time between collisions exceeds the Hubble time ($t_H = H^{-1}$). The turn-off of the DM annihilation reaction and the departure from equilibrium as the temperature falls is the reason for the expression *freeze out.* The temperature at which this transition occurs is called the freeze out temperature, $T_{fo}$. After freeze out, the number density of $\chi$ just dilutes with the expansion of the Universe, $n_\chi \propto a^{-3}$, and remains much larger than the equilibrium value. We illustrate the evolution of $n_\chi$ in Fig. 1.

Let us now come back and justify our assumption of kinetic equilibrium. During freeze out, the rate at which a typical DM particle in the plasma will annihilate with another is approximately $\langle \sigma v \rangle n_\chi$. These annihilation reactions are what allow the DM density to track the following equilibrium value. The annihilation rate should be compared to the rate of reactions that maintain kinetic equilibrium of the DM. One contribution is the elastic scattering of DM with relativistic SM fermions in the plasma, which has a rate of about $\langle \sigma_{el} v \rangle n_f$, where $\sigma_{el}$ is the elastic cross section. Since $n_f \gg n_\chi$ during non-relativistic freeze out and $\sigma_{el} \sim \sigma$ by crossing symmetry, we expect the kinetic equilibrium to be maintained until after freeze out occurs.

We turn next to solving the Boltzmann equation for $n_\chi$ to determine the contribution of $\chi$ to the dark matter density today. As a practical matter, there are three things we need to do:

1. Compute $(\sigma v) 4E_1E_2 = \int (d\Pi_f d\Pi_{\bar{f}})(2\pi)^4 \delta^{(4)}(\ldots)|\widetilde{\mathcal{M}}|^2$. This quantity is Lorentz-invariant, and can therefore always be written as a function of $s = (p_1 + p_2)^2$.
2. Integrate the result over the incoming momenta $p_1$ and $p_2$ weighted by the equilibrium distributions to get $\langle \sigma v \rangle$. The result can be written as a function of $x = m/T$.

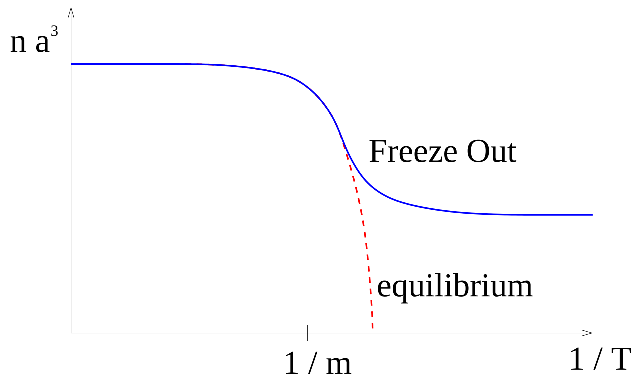


Figure 1: Evolution of $n_\chi a^3$ during thermal freeze out.

3. Put this into Eq. (21) and solve for the density today $n_\chi(t_0)$.

The first two steps present a slight challenge. We usually compute the cross section in a fixed frame (CM or lab in most cases), but our distribution functions are defined relative to the rest frame of the cosmological plasma which usually coincides with neither. The proper way to handle this is to use the Lorentz invariance of $(\sigma v)4E_1E_2$ to re-express the integrals for the thermal average in terms of an integral over $s$ and some other stuff that this quantity does not depend on. The full details can be found in Ref. [3], and the result is straightforward to implement numerically. Instead, we will give an analytic prescription that works well provided freeze out occurs with $\chi$ fairly non-relativistic (which is frequently the case) and $\langle\sigma v\rangle$ a smooth function of $x$. The prescription is:

- Compute $\sigma v$ in the CM frame with initial momenta $p_1 = (m_\chi(1+u^2/2), 0, 0, m_\chi u)$ and $p_2 = (m_\chi(1+u^2/2), 0, 0, -m_\chi u)$.
- Expand the result in powers of $u^2$: $\sigma v = \sigma_0 + \sigma_1 u^2 + \ldots$.
- The effect of thermal averaging is to replace $u^2 \to 3/2x$, where $x = m_\chi/T$: $\langle\sigma v\rangle = \sigma_0 + \sigma_1(3/2x) + \ldots = \sigma_0 + \tilde{\sigma}_1/x + \ldots$.

The result will be correct to $\mathcal{O}(1/x)$. This expansion is related to the orbital angular momentum of the final state; the $\sigma_0$ term is called $s$-wave and the $\sigma_1$ is called $p$-wave.

The third step in finding $n_\chi(t_0)$ reduces to solving a differential equation. For this, it helps to exchange $t$ for $x = m_\chi/T$, and $n_\chi$ for the yield variable $Y_\chi$ defined by

$$Y_\chi := \frac{n_\chi}{s} \ , \tag{22}$$

where $s$ is the entropy density of the thermal plasma,

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \ . \tag{23}$$

Here, $g_{*s}$ is approximately the number of relativistic degrees of freedom in the cosmological plasma. This quantity is extremely useful because the expansion of the Universe conserves the combination $sa^3$ (as long as there are no entropy injections). To exchange $t$ for $x$ as the dependent variable, we use the Hubble equation for radiation domination (for freeze out during this era)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{1}{2t}\right)^2 = g_* \frac{\pi^2}{90}\frac{1}{M_{\rm Pl}^2} T^4 := H^2(T) \ , \tag{24}$$

with $g_*$ the number of relativistic degrees of freedom and $M_{\rm Pl} \simeq 2.4 \times 10^{18}$ GeV the Planck mass.[2] Putting everything together, the rewriting of Eq. (21) in terms of $Y_\chi$ and $x$ is

$$\frac{dY_\chi}{dx} = -\frac{xs}{H(m_\chi)}\langle\sigma v\rangle(Y_\chi^2 - Y_{\chi_{eq}}^2) \ , \tag{25}$$

where $H(m)$ is the value of the Hubble constant at $T = m$ $(x = 1)$ and

$$s = g_{*s}\frac{2\pi^2}{45}m_\chi^3 x^{-3}, \qquad Y_{\chi_{eq}} = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}\frac{g_\chi}{g_{*s}}x^{3/2}e^{-x} \quad (x \gg 1) \ . \tag{26}$$

Everything on both sides of Eq. (25) is a function of $x$ alone.

We are now ready to solve for $Y_\chi(t_0)$, the yield of $\chi$ at the present time. It is straightforward to solve for $Y_\chi$ numerically, and this is what is done in practice. However, we will derive an approximate solution for $Y_\chi$ to get a better intuitive understanding of how freeze out works. The assumption that goes into this approximation is that freeze out occurs when $x = x_{fo} \gg 1$, with $\chi$ being highly non-relativistic. Our strategy will be to estimate $x_{fo}$ and then solve for $Y_\chi$ in the $x \gg x_{fo}$ [1].

We begin by estimating $x_{fo}$, corresponding to the point at which the annihilation process $\chi\chi \to f\bar{f}$ effectively turns off. We will define $x_{fo}$ to be the point where

$$H = \langle\sigma v\rangle n_{\chi_{eq}}\,\kappa \ , \tag{27}$$

where $\kappa$ is a constant of order unity. For $\langle\sigma v\rangle \simeq \sigma_n x^{-n}$, we find

$$x_{fo} \simeq \ln\left[\kappa\sqrt{\frac{90}{8\pi^3}}(g_\chi/g_*^{1/2})m_\chi M_{\rm Pl}\sigma_n\right] - (n+1)\ln\left(\ln\left[\kappa\sqrt{\frac{90}{8\pi^3}}(g_\chi/g_*^{1/2})m_\chi M_{\rm Pl}\sigma_n\right]\right) \ . \tag{28}$$

Note that $x_{fo}$ depends only logarithmically on the unspecified constant $\kappa$ as well as everything else, implying that the sensitivity to the underlying parameters is mild. A good agreement with the numerical solution is found for $\kappa = (n+1)$ [1].

Next, we look for an approximate solution in the region $x \gg x_{fo}$. Since this comes after freeze out, we expect to have $Y_\chi \gg Y_{\chi_{eq}}$. Dropping $Y_{\chi_{eq}}$ terms, Eq. (25) becomes

$$\frac{dY_\chi}{dx} = -\lambda\sigma_n x^{-n-2}Y_\chi^2 \ , \tag{29}$$

[2]Note that my Planck mass is the reduced value $M_{\rm Pl} = 1/\sqrt{8\pi G}$. The non-reduced value $\tilde{M}_{\rm Pl} = 1/\sqrt{G} \simeq 1.2 \times 10^{19}$ GeV is also frequently used.

where $\lambda$ is independent of $x$ and contains all the constants in Eq. (25). This is trivial to solve by integrating between $x_{fo}$ and $x_0 \simeq \infty$ to give

$$Y_\chi(t_0) \simeq \frac{(n+1)\, x_{fo}^{n+1}}{\lambda \sigma_n} \ . \tag{30}$$

Integrating all the way to $x_{fo}$ is a bit of cheat because our approximation starts to break down there, but this cheat is found to still do pretty well compared to the full numerical solution (within 10% or so).

To convert this solution to the relic density $\Omega_\chi h^2$, we need to multiply $Y_\chi(t_0)$ by $m_\chi s_0/\rho_c$. Measurements give $s_0 \simeq 3000\,\mathrm{cm}^{-3}$ and $\rho_c \simeq (1.05\,h^2) \times 10^4\,\mathrm{eV\,cm}^{-3}$. The final result is

$$\Omega_\chi h^2 \simeq (0.23 \times 10^9\ \mathrm{GeV}^{-1})\, \frac{(n+1)\, x_{fo}}{M_{\mathrm{Pl}}\, (g_{*s}/g_*^{1/2})\, \langle \sigma v \rangle} \ , \tag{31}$$

where everything in the denominator is evaluated at $x = x_{fo}$. An important feature of this result is that it depends only very weakly on the mass of the DM particle, logarithmically through $x_{fo}$ (and possibly more directly in $\langle \sigma v \rangle$). The final relic density is also inversely proportional to the annihilation cross section. This isn't surprising – the larger the cross section, the longer the particle density tracks the steeply-falling equlibrium density.

# References


[1] Some of the many nice textbooks on cosmology include:
E. W. Kolb and M. S. Turner, "The Early universe," Front. Phys. **69**, 1 (1990);
S. Dodelson, "Modern cosmology," Amsterdam, Netherlands: Academic Pr. (2003).

[2] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p

[3] P. Gondolo and G. Gelmini, "Cosmic abundances of stable particles: Improved analysis," Nucl. Phys. B **360**, 145 (1991).