Lecture Notes #0: Background

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This note has an overview of the essential background material that you should have seen already in your previous courses. If you have not, please let me know as soon as possible.

1 Need-to-Know Cosmology

Our picture of cosmology is based on (classical) General Relativity and our understanding of elementary particles as described by the Standard Model (SM). To an excellent approximation, the early Universe is a homogeneous plasma of elementary particles. The energy density of this plasma causes the Universe to expand and cool. As it does, interesting things happen that lead to observable consequences today. Some nice textbooks on cosmology can be found in Refs. [1, 2, 3]

Before starting, be warned that throughout this discussion and in the course I will use *natural units* with $\hbar = c = k_B = 1$. Thus, temperatures have units of energy, lengths and times have units of $(\text{energy})^{-1}$, and $M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV $= 1/\sqrt{8\pi G}$. I will say a bit more about converting units below.

1.1 Expansion

Astrophysical measurements of the universe over very large scales indicate that it is approximately homogeneous and isotropic. The most general solution of general relativity with these properties is

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) = -g_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where $a(t)$ is an increasing function of time called the *scale factor*. This solution is sometimes referred to as the Friedmann-Robertson-Walker (FRW) spacetime. The function $a(t)$ is determined by the Einstein equations, and is sourced by the energy density and pressure of the cosmological plasma:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\text{Pl}}^2}\,\rho, \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\text{Pl}}^2}\,(\rho + 3\,p), \tag{3}$$

where ρ is the total energy density of the plasma and p is the pressure. The combination $\dot{a}/a = H$ is sometimes called the *Hubble constant*, even though it usually isn't constant at all.

To solve for $a(t)$, we need to know the time evolution of ρ and p. For this, it is sufficient to know the *equation of state* of the plasma, $p = F(t, \rho)$. In most cases of interest, it takes the linear form

$$p = w(t)\,\rho, \tag{4}$$

The function $w(t)$ depends on time in general, but we can often approximate it by constant. For example, if the universe is dominated by radiation $w = 1/3$, if matter dominates $w = 0$, and for vacuum energy (cosmological constant) $w = -1$. (I will define more precisely what is meant by radiation, matter, and vacuum energy below.) When $w > -1$ is a constant, one can show that

$$\rho \propto a^{-3(1+w)} \ . \tag{5}$$

Plugging this back in to Eq. (2), one finds

$$a(t) \propto \left\{ \begin{array}{ll} t^{1/2} & w = 1/3 \ \ \text{(radiation)} \\ t^{2/3} & w = 0 \ \ \text{(matter)} \end{array} \right. \ . \tag{6}$$

Thus, the energy density of the plasma drives $a(t)$ to increase. In the case of a dominant vacuum energy, $w = -1$, the solution is

$$a(t) \propto e^{H_\Lambda t} \ , \tag{7}$$

with $H_\Lambda = \sqrt{\rho_{vac}/3M_{\rm Pl}}$ is constant.

Using the equations of motion for a point particle in the FRW spacetime, one can show that a particle at rest sitting at the spatial point $\vec{x} = (x, y, z)$ will remain there at later times. This implies that if we have test particles at rest at points $\vec{x}_a$ and $\vec{x}_b$, their physical separations at times t_1 and $t_2 > t_1$ will be

$$\begin{aligned} d_1 &= a(t_1)|\vec{x}_b - \vec{x}_a|, & (8) \\ d_2 &= a(t_2)|\vec{x}_b - \vec{x}_a| = \frac{a(t_2)}{a(t_1)}\, d_1 > d_1. & (9) \end{aligned}$$

We see that the distance between particles at rest increases as $a(t)$ grows. It is in this sense that the universe is said to be expanding. The expansion of the universe also causes photons to redshift, with their wavelengths increasing as $\lambda(t_2)/a(t_2) = \lambda(t_1)/a(t_1)$. Distances to astrophysical objects are frequently quoted in terms of the redshift factor z defined by the relation

$$z := \frac{\lambda(t_0)}{\lambda(t_i)} - 1 = \frac{a(t_0)}{a(t_i)} - 1 \ , \tag{10}$$

where $\lambda(t_0)$ is the wavelength today and $\lambda(t_i)$ is the wavelength when the photon was emitted by the object.

1.2 Expansionary Thermodynamics

The hot plasma driving the expansion of the universe is made up of SM (and possibly other) particles. The average properties of this plasma can be described using standard thermodynamics and statistical mechanics provided the constituents are in thermodynamic equilibrium. Deviations from equilibrium are important for dark matter and the baryon asymmetry, and we will describe how to handle them in the next section.

Radiation is defined to be anything whose mass m_i (and chemical potential μ_i) is much less than the temperature T of the plasma.[1] The energy and number density of the i-th radiation species are given by

$$\begin{aligned} \rho_i &= \frac{\pi^2}{30} g_i \xi_\rho T^4, \\ n_i &= \frac{\zeta(3)}{\pi^2} g_i \xi_n T^3, \end{aligned} \tag{11}$$

where $\xi_\rho = 1\,(7/8)$ for bosons (fermions), $\xi_n = 1\,(3/4)$ for bosons (fermions), and g_i is the number of internal degrees of freedom (like spin) of the particle. The total radiation density is often written as

$$\rho_r = \frac{\pi^2}{30}\, g_*\, T^4, \tag{12}$$

where g_* represents a sum over the $g_i\xi_\rho$ factors of all the relativistic species.

Matter is defined to be any species with $m_i \gg T$, $|\mu_i|$. The energy and number densities of the i-th matter species in thermodynamic equilibrium is

$$\begin{aligned} \rho_i &= m_i\, n_i, \\ n_i &= g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-(m_i-\mu_i)/T}. \end{aligned} \tag{13}$$

This expression shows that the density of *equilibrated* particles heavier than the temperature receives a Boltzmann suppression, and is typically much smaller than the radiation density. We will see below that the dark matter dominates over the radiation density today because it is not in thermodynamic equilibrium.

Vacuum energy corresponds to a non-zero value of the potential for the theory at its (local) minimum. For example, in the potential

$$V(\phi) = \Lambda + \frac{1}{2} b(\phi - \phi_0)^2, \tag{14}$$

the vacuum energy corresponds to the constant $\rho_{vac} = \Lambda$ when $\phi = \phi_0$. In most examples you are likely familiar with, the value of Λ has no physical effect and can be ignored. It is a

[1] Note that the typical momentum of a particle in the plasma is on the order of T, so $T \gg m_i$ implies that particle is highly relativistic on the average.

special (and vexing) feature of GR that the vacuum energy does matter in that it affects the structure of spacetime. Unlike matter or radiation, vacuum energy does not dilute with the expansion of the universe, although its value may change over the course of phase transitions. For this reason it is often called the *cosmological constant*. The value of the vacuum energy today appears to be $\Lambda \sim (10^{-4}\,\text{eV})^4$. It is a mystery why this value is as small as it is since a naïve estimate based on the little we know about quantum gravity suggests a value on the order of $\Lambda \sim M_{\text{Pl}}^4$, a discrepancy of about 10^{120}.

An important feature of the expansion of the universe is that it is *adiabatic*, in the sense that the amount of entropy in a volume $V = a^3$ is constant in time provided all the constituents are in thermodynamic equilibrium. This follows from the Einstein equations for the expansion of the universe and the usual thermodynamic definition of entropy of

$$T\,dS = d(\rho\,V) + p\,dV. \tag{15}$$

The entropy of the universe is usually dominated by relativistic species. When it is, the entropy density is given by

$$s = S/a^3 = \frac{2\pi^2}{45} g_{*S}\,T^3 \tag{16}$$

where g_{*S} is approximately equal to the number of relativistic degrees of freedom of the plasma. From this relation, we see that $T \propto 1/a$ provided g_{*S} is constant.

1.3 A Brief History

The basic picture of the early universe is that it started off as a very hot and very uniform plasma of elementary particles. The energy density of this plasma caused the spacetime of the universe to expand and cool. As the plasma cooled, the particles making it up went through various phase transitions leading to the universe we observe today. We describe here some of the most important transitions as a function of the photon temperature T of the plasma.

- $T \gg$ GeV: *inflation* happens. This gives rise to a very uniform and very hot plasma.
- $T \sim 1$ GeV: QCD phase transition. The cosmological plasma consists of quarks and gluons at temperatures $T \gg 1$ GeV. As the universe cools below $T \sim 1$ GeV, these quarks and gluons coalesce into mesons and baryons.
- $T \lesssim 1$ MeV: big-bang nucleosynthesis (BBN). The baryons (protons and neutrons) in the cosmological plasma combine to initially form Deuterium. These subsequently combine on to form Helium, Lithium, and a handful of other light elements. Heavier elements form much later in stars.
- $T \sim 1\,\text{eV}$: matter-radiation equality and structure formation. Matter replaces the energy of the plasma of light particles (*i.e.* radiation) as the dominant energy component of the universe. Also around this time, the matter in the universe starts to clump together into lumps that will later evolve into galaxies.

- $T \sim 0.1\,\mathrm{eV}$: atom formation (recombination). Free protons and electrons combine to form neutral Hydrogen. This is important because once these stable charged particles form bound states, photons in the cosmological plasma are able to propagate freely. These free photons are what we observe today as the cosmic microwave background radiation (CMB). Since they have propagated freely since decoupling, they provide a snapshot of the state of the universe at this early epoch.

- $T \lesssim 10^{-2}\,\mathrm{eV}$: galaxies and stars start to form.

- $T \sim 10^{-4}\,\mathrm{eV}$: today! This temperature corresponds to a residual CMB photon temperature of about $2.7\,K$. It is uniform up to a spectrum of tiny fluctuations with $\delta T/T \sim 10^{-5}$. The energy density of the vacuum (or something very like it) has recently become the dominant energy component of the universe.

1.4 Handy Numbers

In particle physics, we usually work in natural units where

$$\hbar = c = k_B = 1 \ . \tag{17}$$

This implies that mass, temperature, and momentum have units of energy, while length and time have units of inverse energy. To put the full units back in, you only need to remember three things:

$$\hbar\, c \simeq 197 \text{ MeV fm} \ , \qquad c \simeq 3.00 \times 10^{10}\text{cm/s} \ , \qquad k_B(300\,K) \simeq \frac{1}{40}\,\text{eV} \ . \tag{18}$$

Some conversions that can be derived from this are [1]:

$$1 \text{ GeV} \;=\; 6.58 \times 10^{-25}\text{s} \;=\; 1.97 \times 10^{-14}\text{cm} \;=\; 1.16 \times 10^{13} K \;=\; 1.78 \times 10^{-27}\text{kg} \ . \tag{19}$$

A few other handy things to know are [1]:

$$H_0 \;=\; 100\,h\,\text{km/s/Mpc} \qquad \text{with} \quad h = 0.72 \pm 0.02 \tag{20}$$

$$\rho_c \;=\; 8.10\,h^2 \times 10^{-47}\,\text{GeV}^4 = 1.05\,h^2 \times 10^4\text{eVcm}^{-3} \tag{21}$$

$$s_0 \;=\; 2970 \left(\frac{T_{CMB}}{2.75\text{K}}\right)^3 \text{cm}^{-3} \ . \tag{22}$$

2 Need-to-Know Particle Physics

The second major component of our study of dark matter is particle physics. Dark matter is really interesting for particle physicists because it suggests the existence of a new elementary particle that has not yet been observed directly. A great deal of theoretical and experimental research is focussed on figuring out how to measure the properties of this new particle.

2.1 Quantum Fields and Particles

Quantum Field Theory (QFT) is the primary tool used to describe subatomic particles and the interactions between them. In QFT, quantum mechanics is applied to continuous *field* systems. It might seem conterintuitive to describe seemingly discrete objects like particles by continuous fields. However, in many cases the quantum excitations of field systems behave like independent particles. This observation is borne out by experiments; measurements of elementary particles are described beautifully by QFT.

A QFT is usually specified by the set of fields it contains together with a Lagrangian (density) that encodes how these fields interact with each other. When the interactions described by the Lagrangian are sufficiently weak, each field in the Lagrangian can be identified with a particle species. Furthermore, reactions between the particles can be computed using a set of *Feynman Rules* derived from the Lagrangian.

The structure of the Lagrangian also encodes the underlying symmetries of the theory. For a system to have a symmetry, the form of its Lagrangian should be invariant under a set of field transformations consistent with the symmetry group. This strongly constraints the terms that can appear in the Lagrangian. A particularly important symmetry for relativistic particles (in flat spacetime) is the group of Poincaré transformations, consisting of spacetime translations together with Lorentz transformations. Invariance under translations corresponds to energy-momentum conservation, and the transformation properties of the field variables correspond to the spins of the particles they describe.

Scalar fields describe particles of spin $s = 0$. They do not transform at all under Lorentz transformations in the sense that

$$x \to x' = \Lambda x \ , \qquad \phi(x) \to \phi'(x') = \phi(x) \ , \tag{23}$$

where Λ is the 4×4 Lorentz transformation matrix.[2] The second relation is usually rewritten as

$$\phi'(x) = \phi(\Lambda^{-1}x) \ . \tag{24}$$

The basic Lagrangian for a (real) scalar field $\phi(x)$ is

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - V_{int}(\phi) \ , \tag{25}$$

where $\eta^{\mu\nu}$ is the (inverse) flat-space metric and $V_{int}(\phi)$ contains cubic and higher powers of ϕ and describes the interactions.[3] When the interaction strength is weak, this Lagrangian describes spinless scalar particles of mass m at leading order.

Particles with non-zero spins are described by fields that transform in more complicated ways under Lorentz. In general, when $x \to x' = \Lambda x$, the transformation rule for a field is

$$\Phi'(x) = M(\Lambda)\,\Phi(\Lambda^{-1}x) \ . \tag{26}$$

[2] Explicitly, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

[3] Note that I use the particle physics convention for the flat metric: $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$.

This relation has been written in matrix notation, where $\Phi(x)$ is an n-component column vector and $M(\Lambda)$ is an $n \times n$ matrix corresponding to the Lorentz transformation Λ. These matrices must satisfy $M(1) = \mathbb{I}$ and $M(\Lambda)M(\Lambda') = M(\Lambda\Lambda')$, and they are said to give a *representation* of the Lorentz group.

A simple example is the *vector field* A^μ, transforming as

$$A'^\mu(x) = \Lambda^\mu{}_\nu \, A^\nu(\Lambda^{-1}x) \ . \tag{27}$$

Note that here we have $M(\Lambda) = \Lambda$, which obviously satisfies the multiplication rules of a representation.

The relationship between a particle type and the transformation law of the field that is used to describe it can be a bit complicated. While fields can transform under any representation of Lorentz, particle states must transform under *unitary representations* of Lorentz (as well as spacetime translations), meaning that $M^\dagger(\Lambda) = M^{-1}(\Lambda)$. Analyzing all such representations, the net result is that any *massive* particle type is characterized by its mass and its spin $s = 0, 1/2, 1, 3/2, \ldots$, corresponding to how it behaves under rotations in its rest frame. In this sense, the spin property of elementary particles is seen to emerge from the underlying Lorentz structure. For *massless* particles the situation is a bit different. Such particles can only be in one of two *helicity* states, corresponding to spin along the direction of motion. A familiar example is the photon, which has two independent polarization states.

A popular case is that of a massive particle with $s = 1$, which is usually described by a vector field A^μ. Note, however, that such a field has four independent components, while we only need three for a $s = 1$ particle. The extra degree of freedom corresponds to a particle with $s = 0$, related to field configurations with $A_\mu = \partial_\mu \phi$ for some ϕ. In other words, the vector field describe a particle of $s = 1$ and a particle of $s = 0$.

To project the $s = 0$ piece out and get the $s = 1$ part alone, it is standard to impose the constraint $\partial_\mu A^\mu = 0$. The corresponding Lagrangian is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \ , \tag{28}$$

where m corresponds to the mass of the particle and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{29}$$

is called the *field-strength tensor*. This kinetic term looks a bit funny, but the reason for it is that $F_{\mu\nu} = 0$ for $A_\mu = \partial_\mu \phi$. Thus, the $s = 0$ part has no kinetic term is not dynamical.

A vector field is also used to describe the photon. In this case, the free photon Lagrangian is just the first term of Eq. (28). This is invariant under *gauge transformations*

$$A_\mu \to A_\mu - \partial_\mu \alpha \ , \tag{30}$$

for any function α. Gauge invariance removes the extra, unwanted components of the vector field, and it forbids the mass (second) term present in Eq. (28). The connection with classical electromagnetism is $A^\mu = (\phi, \vec{A})$, the scalar and vector potentials. Using this, it can be shown that the components of $F_{\mu\nu}$ are the gauge-invariant electric and magnetic fields. Note as well that, following our interpretation of classical electrodynamics, two field configurations related by a gauge transformation are taken to be physically equivalent. Invariance under gauge transformations is therefore not a symmetry, but rather an equivalence relation.

We turn next to fermions, with half-integer spins (when they are massive). The simplest field representation is the two-component left-handed Weyl fermion ψ_α transforming as

$$\psi_\alpha \to [M_L(\Lambda)]_\alpha^{\ \beta} \psi_\beta(\Lambda^{-1}x) \ , \tag{31}$$

where $\alpha = 1, 2$ is implicitly summed over here. There is also a right-handed Weyl fermion representation $\bar{\psi}^{\dot{\alpha}}$, related to the left-handed one by

$$\bar{\psi} = i\sigma^2 \, \psi^* \ , \tag{32}$$

where σ^2 is the Pauli matrix. A consistent theory requires both ψ and $\bar{\psi}$ in it. The minimal Lagrangian is

$$\mathscr{L} = \psi^\dagger \, i\bar{\sigma}^\mu \partial_\mu \psi = \psi \, i\sigma^\mu \partial_\mu \bar{\psi} \ , \tag{33}$$

where $\sigma^\mu = (\mathbb{I}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma})$. After imposing the resulting equations of motion, ψ and $\bar{\psi}$ are found to each have one degree of freedom. The interpretation is that ψ corresponds to a massless particle with left-handed helicity and $\bar{\psi}$ to its antiparticle with right-handed helicity.

There are two ways to add a mass term for fermions. When this is done, it is standard (but not always advantageous) to rewrite the two-component spinors are four-component objects. The first type of mass term is called *Dirac* and requires two species of Weyl spinors:

$$\mathscr{L} \supset \psi^\dagger \, i\bar{\sigma}^\mu \partial_\mu \psi + \chi^\dagger \, i\bar{\sigma}^\mu \partial_\mu \chi - M\chi\psi - M\bar{\chi}\bar{\psi} \ . \tag{34}$$

Defining the four-component object Ψ by

$$\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} \ , \tag{35}$$

we can write the terms in Eq. (34)

$$\mathscr{L} \supset \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - M\bar{\Psi}\Psi \ , \tag{36}$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \tag{37}$$

and[4]

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \ . \tag{38}$$

[4]Note that the bar on a two-component object is part of its name, while the bar on a four-component object defined here is a complex-conjugation operation.

The reason for using a four-component object is that it automatically diagonalizes the mass term. After applying the equations of motion, one finds four physical states corresponding to the two spin states of a massive $s = 1/2$ fermion and its antiparticle.

The second kind of mass term is called *Majorana*, and only requires one species of Weyl fermion

$$\mathscr{L} \supset \psi^\dagger\, i\bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} M_M \psi\psi - \frac{1}{2} M_M \bar{\psi}\bar{\psi} \ . \tag{39}$$

To write this in four-component form, let us first define the conjugate of a general four-component fermion Ψ (as in Eq. (35)) by

$$\Psi^c = -i\gamma^2\gamma^0(\bar{\Psi})^t = \begin{pmatrix} \chi \\ \bar{\psi} \end{pmatrix} \ . \tag{40}$$

A four-component *Majorana* fermion Ψ_M is one that satisfies

$$\Psi_M^c = \Psi_M \qquad \Rightarrow \qquad \Psi_M = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \ . \tag{41}$$

In other words, the upper and lower Weyl spinors are constrained to be the same. The kinetic and mass terms of Eq. (39) can now be rewritten as

$$\mathscr{L} \supset \frac{1}{2}\bar{\Psi}_M i\gamma^\mu \partial_\mu \Psi_M - \frac{1}{2} M_M \bar{\Psi}_M \Psi_M \ , \tag{42}$$

where $\Psi_M = \Psi_M^c$. After applying the equations of motion and the Majorana constraint, this system has two independent degrees of freedom corresponding to the two spin states of the massive Majorana fermion. The implication of the Majorana constraint is that a Majorana fermion is its own antiparticle.

To describe the Standard Model (SM) using four-component fermions, it is necessary to define projection matrices. First, let

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{43}$$

Note that $(\gamma^5)^2 = 1$ and $\{\gamma^5, \gamma^\mu\} = 0$. The left- and right-handed projectors are

$$P_L = (1-\gamma^5)/2, \qquad P_R = (1+\gamma^5)/2 \ , \tag{44}$$

and they satisfy

$$1 = (P_L + P_R), \qquad P_L^2 = P_L, \quad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0 \ . \tag{45}$$

Given a four-component fermion Ψ, we define

$$\Psi_L = P_L \Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \ , \qquad \Psi_R = P_R \Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \ , \tag{46}$$

where we have rewritten the components of Eq. (35) as $\psi = \psi_L$ and $\bar{\chi} = \psi_R$.

2.2 The Standard Model

We now have all the pieces we need to assemble the Standard Model (SM) [4, 5, 6, 7]. This theory provides an excellent description of the strong, weak, and electromagnetic forces, and the predictions of the theory are in excellent agreement with a very broad range of experimental measurements. Gravity is not described by the SM since this force is exceedingly weak and almost always neglible in particle physics experiments.

The basis of the SM is gauge invariance under the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Of these factors, $SU(3)_c$ corresponds to the strong force, while $SU(2)_L \times U(1)_Y$ combine to produce the weak and electromagnetic forces. Having fixed the underlying gauge group, all we need to do is to specify the matter content and the vacuum structure. The fermionic matter content comes in three identical copies called *families*. Each family consists of the following representations under $SU(3)_c \times SU(2)_L \times U(1)_Y$:

$$\begin{aligned}
Q_L &= (\mathbf{3}, \mathbf{2}, 1/6) = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
u_R &= (\mathbf{3}, \mathbf{1}, 2/3) \\
d_R &= (\mathbf{3}, \mathbf{1}, -1/3) \\
\\
L_L &= (\mathbf{1}, \mathbf{2}, -1/2) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
e_R &= (\mathbf{1}, \mathbf{1}, -1)
\end{aligned} \tag{47}$$

These are each 2-component fermions that we have written in 4-component notation. Note that these representations do not come in balanced LR and RH pairs, but rather the LH and RH quark and lepton fields have different gauge charges.[5] For Q_L and L_L we have written out the $SU(2)_L$ components explicitly. The Q_L, u_R, and d_R fields transform non-trivially under $SU(3)_c$ and are called *quarks*, while the $SU(3)_c$-neutral L_L and e_R fields are called *leptons*. Aach quark also has three colour components which we have not written out explicitly. In addition to three families of fermions, there is also a single Higgs scalar field

$$\Phi = (\mathbf{1}, \mathbf{2}, 1/2) = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{48}$$

We will write the gauge fields for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ factors as

$$\begin{aligned}
G^a_\mu &\sim (\mathbf{8}, \mathbf{1}, 0) \\
W^p_\mu &\sim (\mathbf{1}, \mathbf{3}, 0) \\
B_\mu &\sim (\mathbf{1}, \mathbf{1}, 0)
\end{aligned} \tag{49}$$

Recall that the **8** of $SU(3)_c$ is the adjoint, as is the **3** of $SU(2)_L$.

[5]Fermions with this property are sometimes said to be *chiral*.

Under $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformations, a given field ψ transforms according to

$$\begin{aligned}\psi_{ir} \to \psi'_{ir} &\equiv U^{(3)}_{ij} U^{(2)}_{rs} U^{(1)} \, \psi_{js} \quad (50)\\ &= (e^{i\alpha^a t^a_{r_c}})_{ij} \, (e^{i\beta^p t^p_{r_L}})_{rs} \, (e^{i\gamma Y} + \ldots) \, \psi_{js}. \\ &= \left[\delta_{ij}\delta_{rs} + i\alpha^a (t^a_{r_c})_{ij}\delta_{rs} + i\delta_{ij}\beta^p (t^p_{r_L})_{rs} + i\delta_{ij}\delta_{rs}\gamma \, Y\right] \psi_{rs}. \quad (51)\end{aligned}$$

That is, ψ carries $SU(3)_c$ (i and j) and $SU(2)_L$ (r and s) indices, and each of these product subgroups acts relative to these indices independently. The quantities α^a, β^p, and γ are the universal group transformation parameters that apply to all representations. When a field transforms as a singlet under $SU(3)_c$ or $SU(2)_L$, the corresponding representation generators vanish and we don't need to include an index for that group on the field. Thus we have

$$Q_L = (Q_L)_{ir}, \quad u_R = (u_R)_i, \quad d_R = (d_R)_i, \quad L_L = (L_L)_r, \quad e_R = (e_R). \quad (52)$$

Woohoo!

The SM Lagrangian takes the form

$$\mathscr{L} = \mathscr{L}_{gauge} + \mathscr{L}_{Higgs} + \mathscr{L}_{Yukawa}. \quad (53)$$

The gauge piece is completely fixed by gauge invariance:

$$\begin{aligned}\mathscr{L}_{gauge} &= -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^p_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 \\ &\quad + \bar{Q}_L i\gamma^\mu D_\mu Q_L + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}_R i\gamma^\mu D_\mu d_R \quad (54)\\ &\quad + \bar{L}_L i\gamma^\mu D_\mu L_L + \bar{e}_R i\gamma^\mu D_\mu e_R,\end{aligned}$$

where each covariant derivative takes the form

$$D_\mu = \partial_\mu + ig_s \, t^a_{r_c} G^a_\mu + ig \, t^p_{r_L} W^p_\mu + ig' \, Y B_\mu, \quad (55)$$

with t^a_{rc} the appropriate $SU(3)_c$ generators for the coresponding rep ($t_{r_c} = 0$ for the trivial rep), $t^p_{r_L}$ the generators for $SU(2)_L$ ($t_{r_L} = 0$ for the trivial rep), and Y is the charge of the field under $U(1)_Y$ and is called *hypercharge*. The Higgs part is

$$\mathscr{L}_{Higgs} = \left| \left(\partial_\mu + ig\,\frac{\sigma^p}{2} W^p_\mu + ig' \frac{1}{2} B_\mu \right) \, \Phi \right|^2 - \left(-\mu^2 |\Phi|^2 + \frac{\lambda}{2} |\Phi|^2 \right). \quad (56)$$

This potential induces spontaneous symmetry breaking whose consequences we will examine presently. Finally, the third set of terms in the SM Lagrangian corresponds to scalar-fermion *Yukawa* interactions of the form

$$\mathscr{L}_{Yukawa} = -y_u \bar{Q}_L \tilde{\Phi} \, u_R - y_d \bar{Q}_L \Phi \, d_R - y_e \bar{L}_L \Phi \, e_R + (h.c.), \quad (57)$$

where $\tilde{\Phi} \equiv i\sigma^2 \Phi = (\phi^{0*}, \, -\phi^{+*})^t$ These interactions are the most general ones we can write (at the renormalizable level) while being consistent with gauge invariance given the charges of Eq. (47). Note that the gauge charges forbid fermion mass terms.

The first step in working out the implications of this Lagrangian is to determine the vacuum structure. The Higgs potential leads to spontaneous symmetry breaking and we can choose a gauge (called the *unitarity gauge*) such that

$$\Phi(x) = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}, \tag{58}$$

where $v = \sqrt{\mu^2/\lambda}$. The remaining h field here is called the Higgs boson. This expectation value has important consequences for the rest of the theory. From the Higgs kinetic term we obtain masses for some of the W^p_μ and B^μ gauge bosons. Inserting this form for the Higgs field into Eq. (57) we also obtain masses for the fermions.

Symmetry breaking in the SM has the same form as the $SU(2) \times U(1)$-invariant theories we considered previously. Applying an arbitrary $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformation to the vacuum state chosen above, we see that this vacuum is invariant under $SU(3)_c$ as well as an Abelian subgroup of $SU(2)_L \times U(1)_Y$. The generator of this subgroup is

$$Q \equiv t^3 + Y. \tag{59}$$

We identify this unbroken subgroup with the $U(1)_{em}$ invariance of electromagnetism, so that the unbroken Q generator defined here corresponds to electric charge. Therefore there should exist a massless gauge boson corresponding to the photon.

To verify this we should construct the gauge boson mass matrix generated by the covariant kinetic term for the Higgs field. This leads to

$$|D_\mu \Phi|^2 \to \frac{1}{2}(\partial h)^2 + \frac{1}{2}\frac{v^2}{2}\left[g^2[(W^1_\mu)^2 + (W^2_\mu)^2] + (-gW^3_\mu + g' B_\mu)^2\right]. \tag{60}$$

From this expression it is clear that two orthogonal linear combinations of W^1_μ and W^2_μ obtain equal masses. It turns out to be convenient to arrange them into the $W^\pm$ vector bosons,

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp i W^2_\mu\right). \tag{61}$$

The reason for this choice is that the two states have charges ± 1 under $U(1)_{em}$. Their equal masses are

$$m_W^2 = \frac{g^2}{2} v^2. \tag{62}$$

For W^3_μ and B_μ we get a squared mass matrix of

$$M^2 = \frac{v^2}{2}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}. \tag{63}$$

As expected, this matrix has a zero eigenvalue corresponding to the photon A_μ. The other linear combination of W^3_μ and B_μ is called the Z^0 vector boson. These mass eigenstates are related to the fields in the original basis by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \tag{64}$$

where the *weak mixing angle* θ_W is defined by

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}\,, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}\,. \tag{65}$$

While the photon is massless, the Z^0 vector boson has mass

$$m_Z^2 = \left(\frac{g^2+g'^2}{2}\right)v^2\,. \tag{66}$$

The longitudinal components of the massive $W^\pm$ and Z^0 vectors account for the missing NGBs from the three broken electroweak generators. Since the new mass eigenstate vector fields we have defined above are related to the original gauge eigenstates by orthogonal transformations, the kinetic terms for the mass eigenstate vectors will also be canonical.

Rewriting the gauge eigenstates in terms of mass eigenstates in the electroweak parts of the matter covariant derivatives we find

$$\begin{aligned} D_\mu &\supset igt^pW^p_\mu + ig'YB_\mu \\ &= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] \\ &\qquad + i(gc_Wt^3 - s_Wg'Y)Z_\mu + i(gs_Wt^3+g'c_WY)A_\mu \\ &= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] + i\bar{g}(t^3-s_W^2Q)\,Z_\mu + ie\,Q\,A_\mu\,. \end{aligned} \tag{67}$$

Along the way we have implicitely defined the couplings

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = gs_W = g'c_W, \qquad \bar{g} = \sqrt{g^2+g'^2}. \tag{68}$$

While the SM has many individual interaction terms, we see that they are all essentially fixed by the values of g, g', and v from the underlying gauge-invariant theory together with the output of the Higgs mechanism. Measurements of the electroweak sector of the SM find that

$$\begin{aligned} m_W &\simeq 80.4\text{ GeV}, \quad m_Z \simeq 91.2\text{ GeV}, \quad v \simeq 174\text{ GeV}, \\ s_W^2 &\simeq 0.23, \quad g \simeq 0.65, \quad g' \simeq 0.45, \quad e^2/4\pi \simeq 1/137. \end{aligned} \tag{69}$$

Note that not all the values of these measurable masses and couplings are independent in the underlying theory. We will see that this allows for very stringent experimental tests of the electroweak sector of the SM.

The remaining pieces of the SM Lagrangian that we have not yet examined are the Yukawa terms. Rewriting the Higgs scalar doublet in terms the new vacuum-friendly field variables, the Yukawa interactions become

$$
\begin{aligned}
-\mathscr{L}_{Yukawa} &= y_u \bar{Q}_L \tilde{\Phi}\, u_R + y_d \bar{Q}_L \Phi\, d_R + y_e \bar{L}_L \Phi\, e_R + (h.c.) \qquad (70)\\
&= y_u\,(v + h/\sqrt{2})\,\bar{u}_L u_R + y_d\,(v + h/\sqrt{2})\,\bar{d}_L d_R + y_e\,(v + h/\sqrt{2})\,\bar{e}_L e_R + (h.c.).
\end{aligned}
$$

This expression consists of Dirac mass terms for the fermions together with fermion-Higgs boson interactions:

$$
m_i = y_i\, v. \tag{71}
$$

In other words, the mass of each SM fermion is proportional to how strongly it couples to the Higgs field.

2.3 Computing Stuff

The main two observable that we will be interested in are scattering cross sections and decay rates. The first step in computing both is to find the corresponding quantum-mechanical amplitude, $\mathcal{M}$. Squaring this quantitity gives a probability density that we can insert into formulas for cross sections and decays.

For $2 \to n$ scattering with two initial particles colliding to make a final state with n particles, let us label the initial four-momenta by p_1 and p_2 and the final four-momenta by p_3-p_{n+2}. The formula for the scattering cross-section is

$$
\sigma = \frac{S}{v}\,\frac{1}{4E_1E_2}\int\frac{d^3p_3}{(2\pi)^3 2E_3}\cdots\int\frac{d^3p_{n+2}}{(2\pi)^3 2E_{n+2}}\;(2\pi)^4\delta^{(4)}(k_1+k_2-\sum_{i=3}^{n+2}p_i)\;|\mathcal{M}|^2\;, \tag{72}
$$

where $v = \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}\,/\,E_1E_2$ is the magnitude of the initial relative velocity as seen in the lab frame, and S is a combinatoric factor equal to one times $1/k!$ for every set of k identical particles in the final state. Derivations of this result can be found in Peskin & Schroeder [5] and Srednicki [8].

The result of Eq. (72) has a lot going on, but its physical content is very simple. First, $|\mathcal{M}|^2$ is the probability density for a single initial state $(p_1{+}p_2)$ to scatter into the specific final state $(p_3{+}\ldots{+}p_{n+2})$. The delta function enforces overall four-momentum conservation. The scattering probability density is then summed over all distinct final states with a relativistic normalization. Collectively, this set of final states is often called the *phase space*. The prefactor before the integrations is a normalization to convert the result for a single initial state to the scattering probability rate per unit incident flux (= number of incident particles per unit area per unit time). At the end of the day, the cross section has units of area. The factor of S accounts for sets of indistinguishable particles.

The quantity of interest for particle decays is the average decay rate Γ. Given an initial sample of N_0 particles at time $t = 0$, the number of particles after time t is

$$
N(t) = N_0 e^{-\Gamma t}\;. \tag{73}
$$

The lifetime τ of a particle species is defined to be

$$\tau = 1/\Gamma \ . \tag{74}$$

Sometimes you will also hear of half-lives, given by $\tau_{1/2} = \tau \ \ln 2$. In natural units, the decay rate has units of mass.

When a particle has more than one distinct decay modes, we also speak of partial decay rates Γ_f, corresponding to the relative probability of decaying in that way. The total decay rate is the sum of the partial rates of all the individual decay channels,

$$\Gamma = \sum_f \Gamma_f = \Gamma \ \sum_f BR_f \ , \tag{75}$$

where $BR_f = \Gamma_f/\Gamma$ is the *branching ratio* to the final state f.

The formula for the partial decay rate of an unstable particle of mass M at rest to decay to a final state containing n particles $(1 \to 2+3+\ldots+n+1)$ is

$$\Gamma(1\to n) = \frac{S}{2M}\int\frac{d^3p_2}{2E_2(2\pi)^3}\cdots\int\frac{d^3p_2}{2E_2(2\pi)^3}\,(2\pi)^4\delta^{(4)}(p_1-\sum_{i=1}^{n+1}p_i)|\mathcal{M}|^2 \ , \tag{76}$$

where $|\mathcal{M}|^2$ is the corresponding $1\to n$ amplitude defined in the same way as for scattering, and S is the symmetry factor.

When some of the initial and final particle states have spin or other internal quantum numbers, the formulas of Eqs. (72,76) apply to transitions from a specific initial spin (and other stuff) state to a specific final spin (and other stuff) state. In many cases of interest, however, the initial state will come from an ensemble with no net spin (and other stuff) polarization and the final spin (and other stuff) state is not measured.[6] The quantities of interest in this case are the "unpolarized" scattering cross section and decay rate. To compute them, just replace $|\mathcal{M}|^2$ in Eqs. (72,76) with the "summed and squared matrix element",

$$\text{“}|\mathcal{M}|^{2\prime\prime} = \frac{1}{N_i}\sum_{\{i\}}\sum_{\{f\}}|\mathcal{M}(\{i\}\to\{f\})|^2 \ , \tag{77}$$

where the sums go over all the possible values of the internal quantum numbers of the initial ($\{i\}$) and final ($\{f\}$) states, and N_i is the total number of initial states. In other words, we should *average over all possible initial states and sum over all possible final states.* Note that we have already done this to some extent in Eqs. (72,76), we sum $|\mathcal{M}(p_1,p_2;p_3,\ldots p_{n+2})|^2$ over all possible final-state momenta. If we had wanted to probability to scatter into a specific value of the outgoing final momentum, we would not integrate over the corresponding phase space.

[6]For example, the scattering of particles in the cosmological plasma will not have a special initial spin configuration, on the average.

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