Lecture Note #2: Thermal Dark Matter Creation

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The existence of a new stable, neutral, massive particle is not enough to explain the observed dark matter. There must also be a mechanism by which the right density of this particle was created in the early Universe. The DM creation mechanism that has received the most attention is *thermal freeze out*, and we will describe it in detail here.

The assumption underlying thermal freeze-out is that the DM particle, (which we will refer to as χ) was once in thermodynamic equilibrium with the hot plasma of SM particles created after inflation. During this period, the Universe was radiation-dominated with a temperature so large that the DM particle was also highly relativistic. As the Universe cooled below the mass of χ, the annihilation reactions $\chi\chi \leftrightarrow SM\,\overline{SM}$ could no longer keep up with the expansion of the Universe and effectively turned off, leaving a *relic density* of χ particles much larger than their equilibrium value. In this case, the χ particle is said to have *frozen out* of the thermal bath, and the leftover density makes up the DM.

1 Equilibrium and Departures from It

The most useful way to describe the properties of the hot soup of particles present in the early Universe is statistical mechanics. Everything we could hope to know about the average properties of the i-th particle species is contained in the *distribution function* $f_i(t, \vec{x}, \vec{p})$. For example, the local number density, energy density, and pressure are

$$\begin{aligned} n_i(t,\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, g_i\, f_i(t,\vec{x},\vec{p}) && (1)\\ \rho_i(t,\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, g_i\, E_i(\vec{p})\, f_i(t,\vec{x},\vec{p}) && (2)\\ p_i(t,\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, g_i\, \frac{\vec{p}^2}{3E_i(\vec{p})}\, f_i(t,\vec{x},\vec{p}) && (3)\\ & && (4) \end{aligned}$$

where g_i is the number of internal degrees of freedom; spins, colours, and such).

You have already encountered distribution functions in thermodynamics. For a (Grand Canonical Ensemble) system in *thermodynamic* equilibrium, we have

$$f_{i_{eq}}(t,\vec{x},\vec{p}) = f_{i_{eq}}(E) = \left[e^{(E-\mu)/T} \mp 1\right]^{-1}\ , \qquad (5)$$

where T is the temperature, μ is the chemical potential, and the minus (plus) sign corresponds to bosons (fermions).

In the early Universe, we are interested in the distributions of elementary particles, which we can usually treat as an unpolarized, weakly-interacting gas. Thus, the state of any single particle is characterized by its 3-momentum $\vec{p}$, with $E = \sqrt{m_i^2 + \vec{p}^2}$. We also know that the early Universe is highly isotropic and homogeneous, which means that we must have $f_i(t, \vec{x}, \vec{p}) = f_i(t, E)$.[1]

Since the Universe is expanding in time, it cannot be in thermodynamic equilibrium in the usual time-independent sense. However, in many cases the time scale of the expansion is very slow compared to the interaction rates of particles in the plasma, and the expansion can be accommodated as an adiabatic change in the temperature (and chemical potentials) of the plasma. As a result, we say that a species in the cosmological plasma is in thermodynamic equilibrium if its distribution function is given by Eq. 5 with only T and μ varying slowly in time. Applying this to the particle species i (and setting chemical potentials to zero which is usually a good approximation), we find the equilibrium number densities

$$n_{i_{eq}} = \begin{cases} \left\{ \begin{matrix} 1 \\ 3/4 \end{matrix} \right\} g_i \frac{\zeta(3)}{\pi^2} T^3; & T \gg m_i \\ g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} & T \ll m_i \end{cases} , \tag{6}$$

where the 1 (3/4) is for bosons (fermions) and $\zeta(3) \simeq 1.202$. At high temperature, we see that $n_i \sim T^3$, while at low temperatures we have $n_i \propto e^{-m_i/T}$ corresponding to the usual Boltzmann suppression of states with $E \simeq m_i \gg T$.

When discussing thermal freeze out of dark matter, we will be interested in computing the departure of the DM distribution function from thermodynamic equilibrium. There will be two aspects to this. Full thermodynamic equilibrium implies both *chemical equilibrium* and *kinetic equilibrium*. Chemical equilibrium means that the number density of the species matches the equilibrium value. Kinetic equilibrium means that the distribution has the same energy dependence as the equilibrium value, or equivalently

$$f_i(t, E) = \xi_i(t) f_{i_{eq}}(E) , \tag{7}$$

where $\xi(t)$ is some function of time alone (but not energy), and the equilibrium distribution depends on time only through the time variation of T and μ. For thermal freeze out of DM, we will see that chemical equilibrium is lost before kinetic equilibrium.

Departure from thermodynamic equilibrium in the distribution f_i is described by a Boltzmann equation of the form

$$L[f_i] = C[f_i; \{f_j\}] . \tag{8}$$

The left side is called the Liouville term and the right side is called the collision term. For a non-relativistic system, the Liouville term is given by

$$\left(\frac{\partial}{\partial t} + \frac{dx_k}{dt} \frac{\partial}{\partial x_k} + \frac{dp_k}{dt} \frac{\partial}{\partial p_k} \right) f_i . \tag{9}$$

We see that it is just a total time derivative. The relativistic generalization is

$$L[f_i] = \left(p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \right) f_i , \tag{10}$$

[1]As discussed before, the tiny spatial variations in the density are very important for structure formation, but they won't play a relevant role in DM freeze out so we will ignore them here.

where $\Gamma^{\alpha}_{\beta\gamma}$ is the Christoffel symbol for the background spacetime. Unlike the Liouville term, the collision term $C[f_i; \{f_j\}]$ is process-dependent and depends on the distributions of other particles present in the plasma. It describes scattering and decay processes that modify the distribution f_i.

Starting from the Boltzmann equation, we can integrate both sides over d^3p_χ to get an equation for the time variation of the density n_χ of the particle species χ in the early Universe. The result is:

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \tilde{C}[f_\chi; \{f_j\}] \ . \tag{11}$$

The $3Hn_\chi$ term on the left describes the dilution of the χ density by the expansion of spacetime; if $\tilde{C} = 0$ we would have $n_\chi \propto a^{-3}$, where $a(t)$ is the expansion factor. The collision term on the right has a contribution from every process in the plasma that changes the number of χ particles. For the process $\chi + a + \ldots + b \leftrightarrow i + \ldots + j$, the contribution to the collision term is given by

$$\begin{aligned}\Delta\tilde{C} &= -\int (d\Pi_\chi d\Pi_a \ldots d\Pi_b)(d\Pi_i \ldots d\Pi_j)(2\pi)^4\delta^{(4)}(p_\chi + p_a + \ldots + p_b - p_i + \ldots + p_j) \\ &\quad \times \frac{1}{S}\left[|\mathcal{M}_{\chi+\ldots+b\to i+\ldots+j}|^2 f_\chi f_a \ldots f_b (1\pm f_i)\ldots(1\pm f_j)\right. \\ &\quad \left. - |\mathcal{M}_{i+\ldots+j\to\chi+\ldots+b}|^2 f_i \ldots f_j (1\pm f_\chi)\ldots(1\pm f_b)\right] \ .\end{aligned} \tag{12}$$

Here $d\Pi_i = g_i\, d^3p_i/(2\pi)^3 2E_i$ is the Lorentz-invariant phase space measure, and $|\mathcal{M}_{I\to F}|^2$ is the squared matrix element for the reaction $I \to F$ averaged over all initial and final degrees of freedom. We also see that the forward reaction $\chi + \ldots + b \to i + \ldots + j$ is weighted by the phase space distributions of all the particles in the initial state and factors of $(1 \pm f_i)$ for each particle in the final state. The sign here is positive if i is a boson and negative if it is a fermion. These factors account for Pauli blocking (reduced final-state phase space for fermions due to Pauli exclusion) or stimulated emission (enhanced phase space by Bose condensation). The symmetry factor S accounts for identical particles, and picks up a factor of $n!$ for every identical species in the initial or final state.

The collision term of Eq. (12) is very complicated, but it can be simplified greatly by making a few reasonable approximations. In most cases of interest we have $f_i \ll 1$, meaning that $(1 \pm f_i) \simeq 1$. It is also usually the case that the effects of CP violation are numerically small. If so, we have $|\mathcal{M}_{\chi+\ldots+b\to i+\ldots+j}|^2 \simeq |\mathcal{M}_{i+\ldots+j\to\chi+\ldots+b}|^2 := |\mathcal{M}|^2$. Plugging this back into Eq. (12), we get

$$\begin{aligned}\Delta\tilde{C} &= -\int (d\Pi_\chi d\Pi_a \ldots d\Pi_b)(d\Pi_i \ldots d\Pi_j)(2\pi)^4\delta^{(4)}(p_\chi + p_a + \ldots + p_b - p_i + \ldots + p_j) \\ &\quad \times \frac{1}{S}|\mathcal{M}|^2(f_\chi \ldots f_b - f_i \ldots f_j) \ .\end{aligned} \tag{13}$$

This is starting to look like the total cross sections for the forward and reverse reactions, integrated over the initial-state phase spaces weighted by the initial-state distribution functions.

2 Thermal Freeze Out of Dark Matter

We now have all the tools we need to study the thermal freeze out of dark matter. To be concrete, we will assume that the DM particle χ is a Majorana fermion (so that $\bar{\chi} = \chi$) and that the only relevant χ-number-changing reaction is $\chi\chi \leftrightarrow f\bar{f}$ for some SM fermion f. The relevant collision term is therefore

$$\Delta\tilde{C} = -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})(d\Pi_f d\Pi_{\bar{f}})(2\pi)^4\delta^{(4)}(\ldots)\,|\mathcal{M}|^2\left(\frac{1}{2}\times 2\right)(f_{\chi 1} f_{\chi 2} - f_f f_{\bar{f}})\ . \tag{14}$$

The factor of $(2 \times 1/2) = 1$ accounts for the fact that the number of χ changes by two units in this reaction, but is cancelled by the symmetry factor for the two identical particles in the initial state.

We are specifically interested in the behaviour of the collision term when $m_f \ll T \lesssim m_\chi$. In this case, we can reliably approximate the equilibrium distributions of f and χ by simple Boltzmann factors ($1/(e^{-m/T} \mp 1) \simeq e^{-m/T}$), and we can assume that the SM fermions f and $\bar{f}$ are in thermodynamic equilibrium. Together with energy conservation in the reaction, this gives

$$f_f f_{\bar{f}} = e^{-(E_f+E_{\bar{f}})/T} = e^{-(E_1+E_2)/T} = f_{1_{eq}} f_{2_{eq}}\ , \tag{15}$$

where $f_{1_{eq}}$ and $f_{2_{eq}}$ are the (Boltzmann) distribution functions that χ_1 and χ_2 would have if they were in thermodynamic equilibrium. Applying this to Eq. (14), we get

$$\begin{aligned}
\Delta\tilde{C} &= -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})\,(f_{\chi 1} f_{\chi 2} - f_{1_{eq}} f_{2_{eq}})\left[\int (d\Pi_f d\Pi_{\bar{f}})(2\pi)^4\delta^{(4)}(\ldots)|\mathcal{M}|^2\right] \\
&= -\int (d\Pi_{\chi_1} d\Pi_{\chi_2})\,(f_{\chi 1} f_{\chi 2} - f_{1_{eq}} f_{2_{eq}})\,(\sigma v)\,(2E_1 2E_2) \\
&= -\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\,g_\chi^2\,(f_{\chi 1} f_{\chi 2} - f_{1_{eq}} f_{2_{eq}})\,(\sigma v)\ .
\end{aligned} \tag{16}$$

In this expression, we have made us of the standard definition of the cross section [2].

We can simplify this last expression even further by making the assumption that the DM particles will remain in kinetic equilibrium throughout the freeze-out process. Using Eq. (7) we get

$$\Delta\tilde{C} = -(n_\chi^2 - n_{\chi_{eq}}^2)\langle\sigma v\rangle\ , \tag{17}$$

where the *thermal average* of an operator $\mathcal{O}(p_1, p_2)$ is defined to be

$$\langle\mathcal{O}\rangle = \frac{1}{n_{eq}^2}\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_1}{(2\pi)^3}\,g_1 g_2\, f_{1_{eq}} f_{2_{eq}}\,\mathcal{O}(p_1, p_2)\ . \tag{18}$$

Note that it is defined with respect to the equilibrium distributions.

Putting everything together, the final form of the Boltzmann equation for the cosmological evolution of the χ density is

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma v\rangle(n_\chi^2 - n_{\chi_{eq}}^2)\ . \tag{19}$$

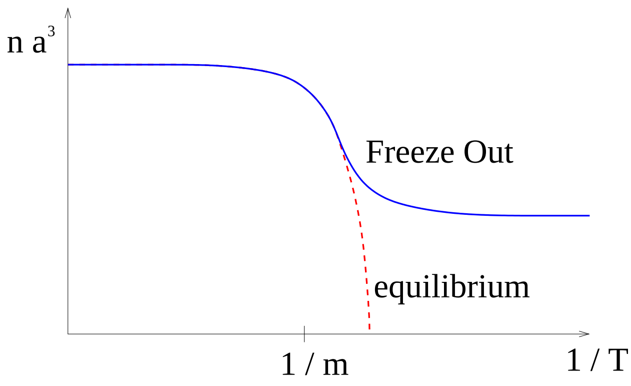

Figure 1: Evolution of $n_\chi a^3$ during thermal freeze out.

Before trying to solve this equation, it's worth thinking a little about how we expect the solutions to look when the cross section is reasonably large. At high temperatures $T \gg m_\chi$, $n_{\chi_{eq}} \sim T^3$ will be relatively large and the collision term on the right will be much larger than the Hubble term. In this case, the number density of χ will be driven to its equilibrium value; $n_\chi \simeq n_{\chi_{eq}}$. On the other hand, as the temperature cools below the mass, $T \lesssim m_\chi$, the equilibrium density of χ becomes exponentially suppressed and the collision term becomes less important. The number density will initially track the equilibrium value until the Hubble term in Eq. (19) becomes larger, at which point the annihilation process effectively turns off. Physically, the number density of χs have become so small that the mean time between collisions exceeds the Hubble time ($t_H = H^{-1}$). The turn-off of the DM annihilation reaction with the falling temperature is the reason for the term "freeze out". After this point, the number density of χ just dilutes with the expansion of the Universe, $n_\chi \propto a^{-3}$, and remains much larger than the equilibrium value. We illustrate the evolution of n_χ in Fig. 1.

We turn now to solving the Boltzmann equation for n_χ to determine the contribution of χ to the dark matter density today. As a practical matter, there are three things we need to do:

1. Compute $(\sigma v) 4E_1 E_2 = \int (d\Pi_f d\Pi_{\bar{f}})(2\pi)^4 \delta^{(4)}(\ldots)|\mathcal{M}|^2$. This quantity is Lorentz-invariant, and can therefore always be written as a function of $s = (p_1 + p_2)^2$.
2. Integrate the result over the incoming momenta p_1 and p_2 weighted by the equilibrium distributions to get $\langle \sigma v \rangle$. The result can be written as a function of $x = m/T$.
3. Put this into Eq. (19) and solve for the density today $n_\chi(t_0)$.

The first two steps present a slight complication. We usually compute the cross section in a fixed frame (CM or lab in most cases), but our distribution functions are defined relative to the rest frame of the cosmological plasma which usually coincides with neither. The proper way to handle this is to make use of the Lorentz invariance of $(\sigma v) 4E_1 E_2$ and to re-express the integrals for the thermal average in terms of an integral over s and some other stuff that this quantity does not depend on. The full details can be found in Ref. [3], and the result is staightforward to implement numerically. Instead, we will provide here an analytic prescription that works pretty well when freeze out occurs with χ very non-relativistic (which is frequently the case). The prescription is:

- Compute σv in the CM frame with initial momenta $p_1 = (m_\chi(1+u^2/2), 0, 0, m_\chi u)$ and $p_2 = (m_\chi(1+u^2/2), 0, 0, -m_\chi u)$.
- Expand the result in powers of u^2: $\sigma v = \sigma_0 + \sigma_1 u^2 + \ldots$.
- The effect of thermal averaging is to replace $u^2 \to 3/2x$, where $x = m_\chi/T$: $\langle\sigma v\rangle = \sigma_0 + \sigma_1(3/2x) + \ldots = \sigma_0 + \tilde{\sigma}_1/x + \ldots$.

The result will be correct to $\mathcal{O}(1/x)$.

The third step in finding $n_\chi(t_0)$ is more straightforward, since all we have to do is solve a differential equation. For this, it helps to exchange t for $x = m_\chi/T$ and n_χ for the yield

$$Y_\chi := \frac{n_\chi}{s} \ , \tag{20}$$

where s is the entropy density of the thermal plasma, given by

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \ . \tag{21}$$

Here, g_{*s} is approximately the number of relativistic degrees of freedom in the cosmological plasma. This quantity is extremely useful because the expansion of the Universe conserves the combination sa^3 (as long as there are no entropy injections). To exchange t for x as the dependent variable, we use the Hubble equation for radiation domination

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{1}{2t}\right)^2 = g_* \frac{\pi^2}{90} \frac{1}{M_{\rm Pl}^2} T^4 := H^2(T) \ , \tag{22}$$

with g_* the number of relativistic degrees of freedom and $M_{\rm Pl} \simeq 2.4 \times 10^{18}$ GeV is the Planck mass.[2] Putting everything together, the rewriting of Eq. (19) in terms of Y_χ and x is

$$\frac{dY_\chi}{dx} = -\frac{xs}{H(m_\chi)} \langle\sigma v\rangle (Y_\chi^2 - Y_{\chi_{eq}}^2) \ , \tag{23}$$

where $H(m)$ is the value of the Hubble constant at $T = m$ ($x = 1$) and

$$s = g_{*s}\frac{2\pi^2}{45} m_\chi^3 x^{-3}, \qquad Y_{\chi_{eq}} = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2} \frac{g_\chi}{g_{*s}} x^{3/2} e^{-x} \quad (x \gg 1) \ . \tag{24}$$

Everything on both sides of Eq. (23) is now written as a function of x alone.

We are now ready to solve for $Y_\chi(t_0)$, the yield of χ at the present time. It is straightforward to solve for Y_χ numerically, and this is what is done in practice. However, we will derive an approximate solution for Y_χ to get a better intuitive understanding of how freeze out works. The assumption that goes into this approximation is that freeze out occurs when

[2] Note that my Planck mass is the reduced value $\tilde{M}_{\rm Pl} = 1/\sqrt{8\pi G}$. The non-reduced value $\tilde{M}_{\rm Pl} = 1/\sqrt{G} \simeq 1.2 \times 10^{19}$ GeV is also frequently used.

$x = x_f \gg 1$, with χ being highly non-relativistic. Our strategy will be to estimate x_f and then solve for Y_χ in the $x \gg x_f$ [1].

We begin by estimating x_f, corresponding to the point at which the annihilation process $\chi\chi \to f\bar{f}$ effectively turns off. We will define x_f to be the point where

$$H = \langle \sigma v \rangle n_{\chi_{eq}} \kappa \,, \tag{25}$$

where κ is a constant of order unity. For $\langle \sigma v \rangle \simeq \sigma_n x^{-n}$, we find

$$x_f \simeq \ln \left[\kappa \sqrt{\frac{90}{8\pi^3}} (g_\chi / g_*^{1/2}) m_\chi M_{\rm Pl} \sigma_n \right] - (n+1) \ln \left(\ln \left[\kappa \sqrt{\frac{90}{8\pi^3}} (g_\chi / g_*^{1/2}) m_\chi M_{\rm Pl} \sigma_n \right] \right) \,. \tag{26}$$

Note that x_f depends only logarithmically on the unspecified constant κ as well as everything else, implying that the sensitivity to the details is pretty weak. A good agreement with the numerical solution is found for $\kappa = (n+1)$ [1].

Next, we look for an approximate solution in the region $x \gg x_f$. Since this comes after freeze out, we expect to have $Y_\chi \gg Y_{\chi_{eq}}$. Dropping $Y_{\chi_{eq}}$ terms, Eq. (23) becomes

$$\frac{d\Delta}{dx} = -\lambda \sigma_n x^{-n-2} \Delta^2 \,, \tag{27}$$

where λ is independent of x. This is trivial to solve by integrating between x_f and $x_0 \simeq \infty$ to give

$$Y_\chi(t_0) \simeq \frac{(n+1)\, x_f^{n+1}}{\lambda \sigma_n} \,. \tag{28}$$

Integrating all the way to x_f is a bit of cheat because our approximation starts to break down there, but this cheat is found to still do pretty well compared to the full numerical solution (within 10% or so).

To convert this solution to the relic density $\Omega_\chi h^2$, we need to multiply $Y_\chi(t_0)$ by $m_\chi s_0/\rho_c$. Measurements give $s_0 \simeq 3000\,\mathrm{cm}^{-3}$ and $\rho_c \simeq (1.05\,h^2) \times 10^4\,\mathrm{eV\,cm}^{-3}$. The final result is

$$\Omega_\chi h^2 \simeq (0.23 \times 10^9\ \mathrm{GeV}^{-1}) \frac{(n+1)\, x_f}{M_{\rm Pl}\, (g_* s / g_*^{1/2})\, \langle \sigma v \rangle} \,, \tag{29}$$

where everything in the denominator is evaluated at $x = x_f$. An important feature of this result is that it depends only very weakly on the mass of the DM particle, logarithmically through x_f (and possibly more directly in $\langle \sigma v \rangle$). The final relic density is also inversely proportional to the annihilation cross section. This isn't surprising - the larger the cross section, the longer the particle density tracks the steeply-falling equlibrium density.

References

[1] Some of the many nice textbooks on cosmology include:
E. W. Kolb and M. S. Turner, "The Early universe," Front. Phys. **69**, 1 (1990);
S. Dodelson, "Modern cosmology," Amsterdam, Netherlands: Academic Pr. (2003).

[2] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p

[3] P. Gondolo and G. Gelmini, "Cosmic abundances of stable particles: Improved analysis," Nucl. Phys. B **360**, 145 (1991).