PSI Dark Matter Homework #2

Due: Apr. 20, 2012

1. Dark matter freeze out, from start to finish.

   a) Starting from the interactions

   $$-\mathscr{L} \supset g_\chi \bar{\chi}\chi\,\phi + g_f \bar{f} f\,\phi \tag{1}$$

   compute the leading contribution to $\langle \sigma v \rangle$ for $\chi\bar{\chi} \to f\bar{f}$ using the non-relativistic prescription described in Note #2. You may treat f as massless but keep the masses of ϕ and χ. You should also account for the fact that ϕ is unstable by including its decay width Γ_ϕ in the propagator: $i/(q^2 - m_\phi^2 + i m_\phi \Gamma_\phi)$.
   Hint: you should have computed the matrix element already in the tutorial session.

   b) Compute x_f and $\Omega_\chi h^2$ using the approximate expressions in Note #2.

   c) Plot your results as a function of the χ mass over the range 10 GeV $< m_\chi <$ 1000 GeV for $g_\chi = g_f = 0.2$ and $m_\phi = 300$ GeV. Take the width of ϕ to be $\Gamma_\phi = m_\phi/50$.

   You should now be equipped to compute the DM relic density of your favourite theory!

2. Non-thermal DM.

   a) Suppose we have a very heavy particle P with mass m_P, decay width $\Gamma_P = \kappa\,(m_P^3/M_{\rm Pl}^2)$, and no relevant annihilation channels. Since this decay width is very small, its lifetime ($\tau_P = 1/\Gamma_P$) will be very long. Assume that the the density of P particles is $n_P = \xi\,T^3$ (with $\xi \ll 1$) when the temperature of the universe is $T = m_P$, and at this time the Universe is dominated by radiation. When does the energy density in non-relativistic P particles become equal to the radiation density? What radiation temperature does this correspond to? What is the energy density ρ_P at this time?

   b) After this time, which we'll call t_P (such that $\rho_P(t_P) = m_P n_P = \rho_R(t_P)$), the density of P particles will be the dominant source of energy in the Universe. The remaining radiation will redshift away due to the expansion induced by P. At the same time, let us assume that P also decays to radiation and to dark matter with relative partial decay widths $\epsilon\Gamma_P$ and $(1-\epsilon)\Gamma_P$ for some $\epsilon \lll 1$. This will inject more radiation and some dark matter into the Universe. These processes are described by the differential equations

   $$\frac{d\rho_P}{dt} + 3H\rho_P = -\Gamma_P \rho_p \tag{2}$$

   $$\frac{d\rho_R}{dt} + 4H\rho_R = +(1-\epsilon)\Gamma_P \rho_P \tag{3}$$

   $$\frac{dn_\chi}{dt} + 3Hn_\chi = +\epsilon\Gamma_P(\rho_P/m_P) - \langle \sigma v \rangle \left(n_\chi^2 - n_{\chi_{eq}}^2\right) \tag{4}$$

   $$H = \left(\frac{\dot{a}}{a}\right) = \sqrt{\frac{8\pi G}{3}}(\rho_P + \rho_R + \rho_\chi)^{1/2} \tag{5}$$

These equations are complicated, but we can solve them analytically by making a few simple approximations. First solve for $(\rho_P a^3)$, starting with the density at $t = t_P$. You should find something proportional to $\exp[-\Gamma_P(t - t_P)]$

c) We are interested in the case $t_P \ll \tau_P^{-1}$. For $t \ll \tau_P = \Gamma_P$, we can approximate the exponential in the solution above by unity: $\exp[...] = 1$. With this approximation, solve for the time dependence of the scale factor $a(t)$ in terms of $a_P = a(t = t_P)$ and t_P in the regime $t_P \leq t \ll \tau_P$. You may also assume that $\rho_{total} = \rho_P$ during this time.

d) Solve also for the energy density in radiation under the same approximations. Figure out which term in your answer is the dominant one for $t \to \tau_P$. What does it correspond to, physically?

e) Solve for the DM density as well, assuming that the annihilation term in the equation is completely negligible. Again, figure out which term in your answer is the dominant one for $t \to \tau_P$ and explain what it corresponds to.

f) Extrapolate your solutions forward in time (under the same approximations), and estimate the time t_{RH} when $\rho_R = \rho_P$ again. How does this compare to τ_P, and what does this say about the approximations made above?
Hint: we are mainly interested in parametric dependences, so don't worry too much about factor of order unity.

g) This point is called *reheating*, and the corresponding temperature is called T_{RH}. What is this temperature in terms of Γ_P?
Hint: lots of stuff cancels out!

h) For $t > t_{RH}$ the Universe reverts to radiation domination. Suppose $T_{RH} \ll T_{fo}$, the freeze-out temperature for χ. In this case, χ does not annihilate significantly after reheating so its number density just dilutes from redshifting once it stops being produced by P decays. Estimate the reheating temperature T_{RH} and the relic density $\Omega_\chi h^2$ today in terms of ϵ for $m_P = 10^5$ GeV, $m_\chi = 100$ GeV, and $\kappa = 10^{-2}$. How does T_{RH} compare to what you would expect for the thermal freeze-out temperature for χ (assuming that it is a WIMP)?

i) A quick approximation to the results above can be obtained by treating P as simply redshifting while it is dominant, and decaying instantaneously to radiation and DM at $t = \tau_P$. Using energy conservation and the fact that each P decay produces an average of ϵ DM particles, find the reheating temperature and DM density in this approximation and compare to your previous results.

The calculations you've done here are almost identical to those you would do for reheating after inflation. In that case, P would correspond to the inflaton field. The main differences are that this intial radiation and DM densities would be nearly zero, and the energy of the P field would come from its oscillation energy (which happens to redshift like matter) rather than its mass.

3. Spin-dependent cross sections.
(Borrowed from notes by P. Salati: http://inspirehep.net/record/776274)

a) The matrix element for χ-nucleus scattering from the spin-dependent AA DM-quark interaction is

$$\mathcal{M} = 2\sqrt{2} G_F \Lambda_N \langle \chi(p_3, s_3) | \, \bar{\chi}\gamma^\mu\gamma^5\chi \, | \chi(p_1, s_1) \rangle \langle N(p_4; J, m_f) | \, S_\mu \, | N(p_2; J, m_i) \rangle \ . \quad (6)$$

Sum this over final states and average it over initial states, treating the nucleus as a non-relativistic system with total spin J (and magnetic states $m = -J, -J+1, \ldots, J$), to show that:

$$\text{“}|\mathcal{M}|^{2\prime\prime} = 8 G_F^2 \Lambda_N^2 \frac{1}{2(2J+1)} \chi^{\mu\nu} N_{\mu\nu} \ , \quad (7)$$

where

$$\chi^{\mu\nu} = tr \left[(\not{p}_3 + m_\chi)\gamma^\mu\gamma^5(\not{p}_1 + m_\chi)\gamma^\nu\gamma^5 \right] \quad (8)$$

and

$$N_{\mu\nu} = \sum_{m_i, m_f} \langle J, m_f | \, S_\mu \, | J, m_i \rangle \langle J, m_i | \, S_\nu \, | J, m_f \rangle \ . \quad (9)$$

b) Work out the trace for χ in the usual way, and simplify it in the lab frame in the extreme non-relativistic limit $v \to 0$ (so that $v \to 0$ and $p_1 = p_3 = (m_\chi, \vec{0})$. You can also drop any terms that will give zero when you contract with the symmetric tensor $N_{\mu\nu}$.
Hint: what could the simplified $\chi_{\mu\nu}$ possibly depend on?

c) Simplify and evaluate $N_{\mu\nu}$. For this, use the fact that $S^\mu \simeq (0, \vec{J})$ in the non-relativistic limit, where $\vec{J}$ is the familiar spin operator.

d) Put everything together to show that “$|\mathcal{M}|^{2\prime\prime} \propto G_F^2 \Lambda_N^2 \, J(J+1)$.