# Tutorial Ideas for # 6

1. **A Scalar in Five Dimensions**
   Consider the theory of a real scalar in $d = 1 + 4 = 5$ dimensions, with the fifth dimension being a flat finite interval $w \in [0, R]$. We will write $x^A = (t, x, y, z, w)$ and $\eta_{AB} = (1, -1, -1, -1, -1)$ (so that $A, B = 0, 1, 2, 3, 5$). The action for the scalar (neglecting interactions) is

$$S = \int d^4x \int_0^R dw \left( \frac{1}{2}\eta^{AB}\partial_A\phi\partial_B\phi - \frac{1}{2}M^2\phi^2 \right) .$$

   To be consistent with the Euler-Lagrange equations of motion, we usually demand either Dirichlet or Neumann boundary conditions (BCs).

   a) Recall that any function $f(w)$ on a finite interval $[0, R]$ with Neumann BCs ($\partial_w f(0) = \partial_w f(R) = 0$) can be expanded as

$$f(w) = \sqrt{\frac{1}{R}}a_0 + \sqrt{\frac{2}{R}}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi w}{R}\right)$$

   With Dirichlet BCs, the expansion is in terms of sines. Show that Dirichlet implies that $f$ is odd under $w \to -w$ and Neumann implies that $f$ is even.

   b) For both the Neumann and Dirichlet cases, compute

$$\int_0^R dw\ f^2(w) \qquad \text{and} \qquad \int_0^R dw\ f(w)\partial_w^2 f(w) .$$

   *Result: usuall KK result.*

   c) Use these results to integrate out the fifth dimension in the action for $\phi = \phi(x, w)$ assuming Neumann boundary conditions. Repeat for Dirichlet boundary conditions.
   *Hint: expand $\phi$ in $w$ with generic $x$-dependent coefficients, then plug into the action and simplify using the orthogonality of sines and cosines.*

   c) In the two cases above, show that the five-dimensional theories can be interpreted as four-dimensional theories containing infinite towers of fields called *Kaluza-Klein modes* and their corresponding profiles (sine or cosine) are called their *wavefunctions*. What are the masses of these fields?

   d) Suppose we add an interaction of the form

$$S \supset -\int d^4x \int_0^R dw\ \lambda^{(5)}\phi^4 .$$

   What is the mass dimension of the coupling $\lambda^{(5)}$? In the Neumann case, compute the non-trivial interactions this induces between a pair of $n = 0$ modes and any number of other modes. What is the range of validity of the theory?

2. **A Fermion in Five Dimensions**
Consider a Dirac fermion $\Psi$ in the same $5d$ spacetime geometry as above.

a) Show that if we define $\Gamma^\mu = \gamma^\mu$ and $\Gamma^5 = i\gamma^5$, we get $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. From this, it can be shown that the four-component fermions give an irreducible representation of the Lorentz group in $d = 1+4$.

b) The basic action for such a fermion is

$$S = \int d^4x \int_0^R dw\ \overline{\Psi} i\Gamma^A \partial_A \Psi\ .$$

Rewrite this in terms of the chiral components $\Psi_{L,R} = P_{L,R}\Psi$.

c) Just like for the scalar, $\Psi$ can be expanded in terms of sines and cosines in the fifth dimension. Furthermore, one can consider different boundary conditions for each of the chiral components. Write the action in terms of these KK modes assuming that $\Psi_L$ has Neumann BCs and $\Psi_R$ has Dirichlet. What are the KK mode masses?
*Result: unpaired $n = 0$ mode with zero mass.*

d) Fermion representations in 5d are non-chiral, but this trick with different BC's gives a way to get SM-like chiral fermions in the 4d reduced theory. Consider a theory with two 5d Dirac fermions $\Psi$ and $\chi$ such that $\Psi_L$ and $\chi_R$ are Neumann, $\Psi_R$ and $\chi_L$ are Dirichlet, and the 5d Lagrangian contains the term

$$\mathscr{L}_5 \supset -M\overline{\Psi}_L \chi_R - M\overline{\chi}_R \Psi_L\ .$$

What are the KK mode masses in this case?
*Result: zero modes are chiral, but have a mass $M$.*

e) In many constructions of this type, an additional requirement that is imposed is that the Lagrangian be even under $w \to -w$. Show that with the BCs specified above, this condition forbids a mass term of the form $\overline{\Psi}\Psi$ but does permit the kinetic terms and the mixed mass term of d).

3. **A Vector in Five Dimensions**
Consider a vector field $A^M(x, w)$ in this setup. It now has five components.

a) We would like for the theory of the vector to have a gauge invariance under $A_M \to A_M - \partial_M \alpha$ for any $\alpha(x, w)$. We would like $\alpha$ to have a definite parity (even or odd) under $w \to -w$. If $\alpha$ is even, what does this imply for the BCs of the $A_\mu$ and $A_5$ components of $A_M$ if the gauge transformation law is to be self-consistent?

b) The basic action for the vector is

$$S = \int d^4x \int_0^R dw\ -\frac{1}{4} F_{MN} F^{MN}\ .$$

Write this in terms of the $A_\mu$ and $A_5$ components of $A_M$.

c) Show that it is always possible to choose a gauge such that $A^5 = 0$. What is the Lagrangian in this gauge? Find the masses of the vector KK modes for $A_\mu$ even.