Tutorial Ideas for # 5

1. **Charged Pions**
   Let's examine some of the claims we made about pions from two-flavour QCD:

   a) We stated that the QED generator was
   $$Q = t^3_L + t^3_R + \mathbb{I}/6 \ ,$$
   Acting on the sigma field Σ, this means that we set $\alpha^a_L = \delta^{a3}\alpha = \alpha^a_R$ and $\alpha_{em} = \alpha/6$ for some transformation parameter α so that finite QED transformations take the form
   $$\Sigma \to e^{i\alpha t^3} e^{i\alpha/6}\, \Sigma\, e^{-i\alpha t^3} e^{-i\alpha/6} \ .$$
   Work out what this implies for the transformation properties of the Π^a fields for infinitesimal α.

   b) Define $\pi^0 = \Pi^3$ and $\pi^\pm = (\Pi^1 \mp i\Pi^2)/\sqrt{N}$.
      i) Express $\Pi^a t^a$ in terms of π^0, $\pi^\pm$ and combinations of the t^a.
      ii) Show that $[t^3, t^1 \mp it^2] = (\mp)(t^1 \mp it^2)$.
      iii) Use these results to figure out the infinitesimal QED transformations of the π^0 and $\pi^\pm$ fields.

   c) Write out the kinetic terms in terms of π^0, $\pi^\pm$. What value of N gives a canonical normalization for the charged pions?
   Hint: "canonical" is $(\partial\phi)^2/2$ for a real scalar and $\partial\Phi^*\!\cdot\partial\Phi$ for a complex scalar.

2. **Gauging Electromagnetism**
   Electromagnetism is conserved below Λ_{QCD}, and we would like to preserve this invariance within the effective theory.

   a) Recall that for a field ψ with $U(1)_{em}$ charge q, the transformations
   $$\psi \to e^{iq\alpha(x)}\psi = (1 + iq\alpha + \ldots)\psi \ , \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha \ ,$$
   imply that $D_\mu\psi = (\partial_\mu + iqeA_\mu)\psi \to e^{iq\alpha}D_\mu\psi$. By analogy to this form, and using the infinitesimal EM transformation on Σ you found above, construct a covariant derivative for Σ and use this to build the leading term in the chiral Lagrangian that is $SU(2)_L \times SU(2)_R$-invariant. Check that it is indeed covariant to leading non-trivial order in $1/f$.
   Hint: note the connection between the infinitesimal transformation of the field and the covariant derivative.
   Result: the infinitesimal transformation of Σ is $\Sigma \to \Sigma + i\alpha[t^3, \Sigma]$. Following the form on the basic complex field ψ, we can get the covariant derivative by taking the infinitesimal variation of the field and replacing it by eA_μ. This gives $D_\mu\Sigma = \partial_\mu\Sigma + ieA_\mu[t^3, \Sigma]$.

b) Recall as well that if the Lagrangian of a theory is invariant under an infinitisimal global field transformation $\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \alpha\Delta\phi_i(x)$, the corresponding *Noether current* is conserved,

$$j^\mu = \sum_i \frac{\partial \mathscr{L}}{\partial(\partial\phi_i)}\Delta\phi_i \ .$$

Find the conserved current for global EM transformations in the two-flavour QCD theory and to leading non-trivial order in the corresponding chiral perturbation theory.
Result:
QCD: $j^\mu = (2/3)\overline{u}\gamma^\mu u + (-1/3)\overline{d}\gamma^\mu d$.
χPT*:* $j^\mu = i\pi^+\partial^\mu\pi^- - i\pi^-\partial^\mu\pi^+$ *– my notes have a typo.*

c) Show that gauging only a subgroup of the total global flavour group explicitly breaks the invariance under flavour. Despite this breaking, we can still formulate a theory that is consistent with the full flavour symmetries in the limit of $e \to 0$.
Result: apply an infinitesimal t_L^1 *or* t_L^2 *rotation to the theory and show that the extra pieces in the covariant derivative mess up the invariance.*

3. **Pion Masses**
Expand out the term

$$\mathscr{L} \supset \frac{1}{2}\tilde{\Lambda}^3 tr(M\Sigma) + h.c \ ,$$

where $M = diag(m_u, m_d)$, and derive mass terms for the pions. What are their values?
Result: the linear term is imaginary and vanishes, so the first non-trivial term in this expansion is quadratic in the Π *fields. Computing the traces explicitly, one gets* $tr(Mt^at^b) = \delta^{ab}(m_u + m_d)/2$. *This yields masses* $m_\Pi^2 = \tilde{\Lambda}^3(m_u + m_d)/f^2$.