Notes #4: Extra Dimensions

David Morrissey
March 10, 2016

A third approach to the electroweak hierarchy problem is to postulate that the fundamental Planck scale is not much larger than the electroweak scale. In these scenarios based on this approach, the strength of gravity is diluted in a way that makes it appear to be much weaker (to us) than it really is. In particular, the true scale of quantum gravity is $M_* \ll M_{\rm Pl}$. If $M_* \sim 4\pi m_W$, our quantum field theoretic description of elementary particles is unlikely to be valid at energies above M_*, and thus there is no hierarchy problem provided M_* is not too much larger than the weak scale.

The known mechanisms for diluting the apparent strength of gravity typically make use of extra spacelike dimensions.[1] Such extra dimensions appear to be an essential component of string theories [1, 2, 3], and they have been studied in various other contexts as well [4]. In the *Large Extra Dimensions* (LED) scenario, the strength of gravity we see is reduced by a factor of the volume of the extra dimensions [5, 6, 7]. With a *Warped Extra Dimenson*, gravity appears to be extremely weak because it is localized away from us in the extra dimension [8, 9]. We will discuss both of these scenarios in these notes, as well as an intriguing connection between warped scenarios and strong coupling in four dimensions.

1 Large Extra Dimensions

Suppose we have N extra dimensions and let M_* be the fundamental Planck scale in the full $d = (4 + N)$-dimensional theory. Consider the gravitational potential Φ in the weak (Newtonian) limit. It satisfies

$$\vec{\nabla}^2\Phi \;\sim\; \frac{1}{M_*^{2+N}}\,\rho\;, \tag{1}$$

where ρ is the local energy density. For a pair of static point masses separated by a distance r, this leads to a gravitational force of

$$F(r) \;\sim\; \frac{1}{M_*^{2+N}}\frac{m_1 m_2}{r^{2+N}}\;. \tag{2}$$

In contrast to $d = 4$, the gravitational flux lines can now spread out in more ways leading to a faster decrease of the force with distance. This is clearly inconsistent with the observed behaviour of gravity.

Let us now modify this picture by taking the N extra dimensions to all be periodic with radius R. Thus, for every extra-dimensional coordinate w we have $w^a \sim w^a + 2\pi R$, $a = 1, 2, \ldots N$. The gravitational force law takes the same form as before at short distances

[1] Timelike extra dimensions lead to challenges with causality and such.

$r \ll R$. However, for distances large compared to the radius of the extra dimensions we have

$$F(r) \sim \frac{1}{M_*^{2+N}} \frac{m_1 m_2}{r^2 (2\pi R)^N} \qquad (r \gg R) . \tag{3}$$

The extent to which the flux lines can spread is now limited by the finite size of the extra dimensions. Matching this expression to the previous one, we find that

$$M_{\rm Pl}^2 = (2\pi R)^n M_*^{2+N} = V_N M_*^{2+N} , \tag{4}$$

where V_N is the total volume of the compact extra dimensions.

Together, Eqs. (3,4) show how the strength of the gravitational force we observe can be diluted by the volume of extra dimensions. The idea of Refs. [5, 6, 7] was to use this dilution to recast the hierarchy problem by making R large enough that $M_* \sim$ TeV.[2] For n extra dimensions of equal size, the required radius R is

$$2\pi R \simeq 10^{32/n} 10^{-17}\text{cm} \sim \left\{ \begin{array}{ccccc} 10^{15}\,\text{cm} & (R^{-1} \sim ...) & ; & N=1 \\ 1\,\text{mm} & (R^{-1} \sim 10^{-13}\ \text{GeV}) & ; & N=2 \\ 1\,\mu\text{m} & (R^{-1} \sim 10^{-8}\ \text{GeV}) & ; & N=3 \\ 10\,\text{fm} & (R^{-1} \sim 10^{-2}\ \text{GeV}) & ; & N=6 \end{array} \right. . \tag{5}$$

These radii are very large compared to typical particle physics scales. For this reason, scenarios of this type are referred to as *large extra dimensions* (LED) or ADD after the original authors [5, 6, 7]. In this section we will discuss the implications of LED models. We will go over the new particles they predict, and we will examine the extent to which they are constrained by existing data and how they may be probed in the future.

1.1 Kaluza-Klein Modes

To investigate the implications of LED, we will need to study quantum fields defined in more than four dimensions. When the extra dimensions are compact, a single d-dimensional field can be reduced to a set of four-dimensional fields called *Kaluza-Klein* (KK) modes. We will show how such KK modes arise in this subsection within a simple scalar model.

Let us write $x^M = (x^\mu, w^a)$ for the $d = 4+N$ spacetime coordinates with $M = 0, 1, 2, \ldots, 3+N$ and $a = 1, \ldots N$. We denote the full d-dimensional metric by G_{MN} with the flat space limit being $\eta_{MN} = diag(+1, -1, \ldots, -1)$. Note as well that when there is only one extra dimension, it is common practice to use the indices $M = 0, 1, 2, 3, 5$.

Consider now a real scalar field $\Phi(x, w)$ in $d = 5$ dimensions with the extra dimension periodic with radius R, $w \sim w + 2\pi R$. The basic action in a flat background is

$$S = \int d^4x \int_0^{2\pi R} dw \left[\frac{1}{2} \eta^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{2} m^2 \Phi^2 \right] . \tag{6}$$

[2] This isn't quite a solution to the hierarchy problem unless a mechanism to fix the radii of the extra dimensions is specified [10].

Given the geometry of the extra dimension, the field should also be periodic: $\Phi(x, w+2\pi R) = \Phi(x, w)$. This implies that the field can be expanded in a set of orthonormal basis functions $\{f_n\}$ according to

$$\Phi(x, w) = \sum_{n\in\mathbb{Z}} f_n(w)\, \phi^{(n)}(x) \ , \tag{7}$$

where the coefficients $\phi^{(n)}(x)$ depend only on the usual spacetime dimensions and

$$\int_0^{2\pi R} dw\, f_m(w)\, f_n(w) = \delta_{mn} \ . \tag{8}$$

The appropriate basis functions in this case are

$$f_n(w) = \frac{1}{\sqrt{2\pi R}} e^{inw/R} \ , \qquad n \in \mathbb{Z} \ . \tag{9}$$

These functions are clearly periodic and orthonormal. Since $\Phi(x, w)$ is real-valued, we must also have ${\phi^{(n)}}^\dagger(x) = \phi^{(-n)}(x)$. Note as well that the basis functions in the extra dimension(s) are often called *wavefunctions*.

Plugging the expansion of Eq. (7) back into the action and using orthonormality, we get

$$\begin{aligned} S \ &= \ \int d^4x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^{(0)}\partial_\nu\phi^{(0)} - \frac{1}{2}m^2{\phi^{(0)}}^2 \right. \\ &\quad + \left. \sum_{n=1}^{\infty} \left[\eta^{\mu\nu}\partial_\mu{\phi^{(n)}}^\dagger\partial_\nu\phi^{(n)} - (m^2 + \frac{n^2}{R^2})|\phi^{(n)}|^2 \right] \right) \ . \end{aligned} \tag{10}$$

This is a four-dimensional theory containing a single real scalar $\phi^{(0)}$ of mass m and a tower of complex scalars $\phi^{(n)}$ with masses $m_n = \sqrt{m^2 + n^2/R^2}$, $n = 1, 2, \ldots$. The lightest state is called a *zero mode*, while the $n \geq 1$ fields are the KK modes.

The KK modes are massive even in the limit of $m^2 \to 0$, and the extra contributions to their masses can be related to a quantized momentum in the fifth dimension. This can be seen by noting that the expansion of Eq. (7) is just a Fourier transform of the extra dimension for a periodic space. A further confirmation of this property can be found by examining a non-trivial interaction:

$$\begin{aligned} S \ &\to \ S_{free} - \int d^4x \int dw\, \lambda_5 \Phi^4 \\ &= \ S_{free} - \int d^4x\, \frac{\lambda_5}{2\pi R} \sum_{k,l,m,n} \phi^{(k)}\phi^{(l)}\phi^{(m)}\phi^{(n)}\, \delta_{k+l+m+n,\,0} \ . \end{aligned} \tag{11}$$

This interaction conserves momentum in the extra dimension at the vertex. Note as well that the theory is non-renormalizable since λ_5 has a mass dimension of minus one, such that $\lambda_5/2\pi R$ is dimensionless. Even so, the theory is still predictive for $E \ll \lambda_5^{-1} \lesssim 4\pi/R$, where the second inequality arises from the requirement that the effective coupling of the KK modes is perturbative.

It is straightforward to generalize the KK expansion to multiple compact extra dimensions. For example, with N periodic extra dimensions of radius R, the basis functions would be

$$f_{\vec{n}}(\vec{w}) = \frac{1}{(2\pi R)^{N/2}}\, e^{i\vec{n}\cdot\vec{w}/R} \ , \qquad\qquad \vec{n} \in \mathbb{Z}_N \ , \tag{12}$$

where $\vec{w} = (w^1, \ldots, w^N)$ and $\vec{n} = (n^1, \ldots, n^N)$. The corresponding KK masses in this case are $m_{\vec{n}}^2 = m^2 + \vec{n}^2/R^2$.

1.2 Gravitons in LED

Let us turn next to realistic particle theories in LED. We will assume there are N periodic extra dimensions of radius R. If we were to put the SM in the full extra dimensional space, every SM particle would have KK modes separated in mass by $1/R$. Since $1/R$ is very small by particle physics standards in LED theories (for N not too large), such KK modes would already have been observed if they existed. Instead, the SM fields are assumed to be restricted to a four-dimensional subsurface of the full spacetime. This is found to occur in many cases in string theory, where fields can be confined to dynamical subsurfaces called *branes* [11, 12]. The only field that propagates within the full spacetime in LED models is the graviton, which therefore develops KK modes.

The graviton in LED emerges from expanding the metric around a background spacetime, which we will assume to be flat,

$$G_{AB} \ = \ \eta_{AB} + h_{AB}/M_*^{1+N/2} \ , \tag{13}$$

where the factors of M_* ensure that h_{AB} has the correct dimensions for a bosonic field in $d = 4 + N$ dimensions. The corresponding graviton action is

$$S_{grav} = \frac{M_*^{N+2}}{2} \int d^4x \int d^n w\, \sqrt{|G|}\, R^{(d)} \ , \tag{14}$$

where $R^{(d)}$ is the d-dimensional Ricci tensor built from G_{AB}. Expanding out the metric in this expression produces kinetic terms for h_{AB} as well as self-interactions. We can rewrite the action in terms of a set of KK modes by expanding in terms of basis functions,

$$h_{AB}(x,w) = \sum_{\vec{n}} \frac{1}{(2\pi R)^{N/2}}\, h_{AB}^{(\vec{n})}(x)\, e^{i\,\vec{n}\cdot\vec{w}/R} \ . \tag{15}$$

Putting this back into Eq. (14), the integration over the extra dimensions can be performed explicitly.

Compared to the simple scalar theory presented above, there is a new twist to the graviton [15, 16, 17, 18]. It has multiple components related to the spacetime indices it carries, and these components transform in different ways under the four-dimensional Lorentz subgroup of the full spacetime coordinate invariance. Starting with $h_{AB}(x,w)$, it is a two-index symmetric tensor with real entries. This would give $d(d+1)/2$ degrees of freedom, but some of these turn out to be related by the underlying invariance under general coordinate transformations.

$$G_{AB} \to G'_{PQ} = \frac{\partial x^A}{\partial y^P}\frac{\partial x^B}{\partial y^Q} G_{AB} \ . \tag{16}$$

For an infinitesimal coordinate transformation $y^A = x^A + \xi^A$, this implies that

$$h_{AB} \to h'_{PQ} = h_{PQ} - \partial_P \xi_Q - \partial_Q \xi_P \ . \tag{17}$$

We can use this invariance to project out the redundant components. A popular choice is harmonic gauge

$$\partial_A h^A{}_B = \frac{1}{2}\partial_B h^A{}_A \ , \tag{18}$$

which imposes d conditions. This gauge still allows residual transformations with $\partial^2 \xi_A = 0$, and this fixes an additional d components. Together, we have a total of $d(d+1)/2 - 2d = d(d-3)/2 = (N+1)(N+4)/2$ independent components.

The next step is to decompose the KK modes of the graviton into quantities with well-defined transformations under the four-dimensional Lorentz subgroup. At each KK level, we have [16, 17, 18]

$$h^{(\vec{n})}_{AB} \ \to \ \left\{ \begin{array}{l} h^{(\vec{n})}_{\mu\nu} \\ h^{(\vec{n})}_{\mu a} = V^{(\vec{n})}_{\mu a} \\ h^{(\vec{n})}_{ab} = S^{(\vec{n})}_{ab} \end{array} \right. \ . \tag{19}$$

The zero modes $\vec{n} = \vec{0}$ are massless and independent of $\vec{w}$. Counting them, we have two degrees of freedom for the massless graviton, $2N$ degrees of freedom for the massless vectors, and $N(N+1)/2$ for the massless scalars to give $(N^2+5N+4)/2$ in total, as expected. The KK modes have masses $m^2_{\vec{n}} = \sqrt{\vec{n}^2/R^2}$. At level $\vec{n}$, there is one massive graviton with five degrees of freedom, $(N-1)$ massive vectors with three degrees of freedom each, and $N(N+1)/2 - N$ massive scalars for a total of (N^2+4N+5). In counting these, we have made use of the constraints implied by general coordinate invariance which translate into the conditions

$$n^a V^{(\vec{n})}_{\mu a} = 0 \ , \qquad n^a S^{(\vec{n})}_{ab} = 0 \ . \tag{20}$$

These can also be understood in that at each KK level the massless graviton eats a massless vector and a massless scalar to get a mass, while the remaining $(N-1)$ massless vectors each eat a scalar.

Having classified all the modes, we would like to figure out how they couple to the SM fields. Recall that the SM is assumed to be confined to a four-dimensional brane, and we can choose its location to be $w^a = 0$. The coupling to gravity is the usual minimal form, but now with an explicit localization,

$$S_{SM} = \int d^4x \int d^n w \, \sqrt{|G|} \, \mathscr{L}_{SM} \, \delta^{(n)}(\vec{w} - \vec{0}) \tag{21}$$

$$= \int d^4x \, \sqrt{|g|} \, \mathscr{L}_{SM} \tag{22}$$

where $g_{\mu\nu}(x) = G_{\mu\nu}(x, w = 0)$ is the *induced metric* on the brane. Expanding this out, one obtains couplings of the form [16, 17]

$$S_{SM} \supset -\int d^4x \, \frac{1}{M_*^{1+N/2}\sqrt{V_N}} \left[T^{\mu\nu} \sum_{\vec{n}} h^{(\vec{n})}_{\mu\nu} - \kappa \, T^{\mu}_{\ \mu} \sum_{\vec{n}} S^{(\vec{n})\ a}_{a} \right] , \tag{23}$$

where κ is an N-dependent constant of order one and

$$T^{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S_{SM}}{\delta g_{\mu\nu}} \tag{24}$$

is the energy-momentum tensor of the SM. The first term reproduces the usual coupling of the massless graviton (zero mode) to the SM once we identify $M_*^{N+2} V_N = M_{\rm Pl}^2$, as well as couplings of the SM to the KK gravitons. The second term connects the SM to a specific linear combination of the scalars called the *radion* and its KK modes. Note that none of the other components of the graviton zero or KK modes couple to the SM. For this reason, they can (mostly) be ignored, and the only graviton excitations that need to be considered are the massless $h^{(0)}_{\mu\nu}$ and $r^{(0)} = S^{(\vec{0})\ a}_{a}$ zero modes, and the massive $h^{(\vec{n})}_{\mu\nu}$ and $r^{(0)} = S^{(\vec{n})\ a}_{a}$ KK modes.

The existence of a massless radion zero mode is a problem since it would modify gravity at long distances. In this context, the masslessness of the radion correponds to a flat "potential" for expanding or shrinking the radius R of the extra dimensions. A stabilization mechanism is needed to fix this, and the massless radion is expected to develop a mass as a result [10]. In fact, a severe challenge to LED models is that it is very difficult to stabilize the radion(s) at a large enough radius to give $M_* \sim$ TeV. From this point of view, the LED construction has traded the old electroweak hierarchy problem for a new one. Still, a different perspective on any challenge can be useful, and we will see that radion stabilization can be arranged in theories with a warped extra dimension to be discussed below.

1.3 Experimental Tests and Constraints

The new graviton KK modes predicted by LED models are constrained and are being looked for in a number of ways. The strongest bounds typically come from deviations from $1/r^2$ gravity and modifications to stellar evolution, while searches for the new KK states are underway at the LHC. Limits on LED are usually quoted in terms of a lower bound on M_* for a given number of extra dimensions N. Recall that we want $M_* \lesssim$ TeV for this scenario to address the electroweak hierarchy problem. A more detailed discussion can be found in Ref. [18].

Light graviton (or radion) KK modes can modify the $1/r^2$ behaviour of the gravitational force at short distances, $r \lesssim R$. Such deviations have been investigated, and it is standard practice to parametrize them according to

$$V(r) = -\frac{1}{8\pi M_{\rm Pl}^2}\frac{m_1 m_2}{r}\left(1+\alpha e^{-r/\lambda}\right) \ , \tag{25}$$

where λ and α are dimensionless parameters. For the first deviations from massive KK gravitons, we expect $\alpha = 1$ and $\lambda = R$. The current limit for $\alpha = 1$ is $\lambda \leq 44\,\mu$m [19]. This rules out $N = 1$, and forces $M_* \gtrsim 1.4$ TeV for $N = 2$, but does not provide a useful constraint for $N \geq 3$.

The LED theory with KK modes at $M_* \sim$ TeV only works as an effective theory valid at energies below $E \lesssim M_*$. New physics above this scale is unknown, but will include quantum gravity and is expected to generate non-renormalizable operators involving SM fields such as

$$-\mathscr{L} \ \supset \ \frac{1}{M_*^2}(\bar{f}_1\Gamma f_2)(\bar{f}_3\Gamma' f_4) \ , \tag{26}$$

where $f_{1,2,3,4}$ are SM fermions, and Γ and Γ' are Dirac structures. If such operators violate baryon and lepton number, limits on nucleon decay force $M_* \gtrsim 10^{16}$ GeV. If they generate new flavour mixing or CP violation, $M_* \gtrsim 10^6$ GeV. And even if they respect all the symmetries of the SM, precision electroweak constraints limit $M_* \gtrsim 10$ TeV. The suppression of these operators cannot be addressed within the LED effective theory, but they are a cause for concern.

Collider experiments have also searched for the direct production of KK modes. These modes would typically escape the collider detectors without leaving a trace, and they would therefore contribute to missing energy. The coupling of the SM to any single mode is tiny, suppressed by $M_{\rm Pl}$, but there are many closely-spaced modes to sum over and this makes the effective cross section potentially observable. Once all the accessible modes are added up, the cross section scales like $1/M_*^2$ rather than $1/M_{\rm Pl}^2$. Searches at the LHC often focus on events with a single hard jet and a large amount of missing energy, which can be generated when a KK gluon is radiated off a gluon. Current LHC data limits $M_* \gtrsim 5.5$, 4.3, 3.2 TeV for $N = 2, 4, 6$ as shown in Fig. 1 [20, 21].

The many light KK modes predicted by LED theories can also contribute to astrophysical processes. In particular, they can cause supernovae to cool more quickly that they would otherwise. This can occur through the production of KK gravitons in the very hot interior of a supernova as it is starting to explode, where the temperature can approach $T \simeq 50$ MeV. The gravitons can then escape the supernova envelope and cause it to cool faster than is observed. The strongest limits come from observations of SN1987A, and give $M_* \gtrsim 50$, 4, 1 TeV for $N = 2, 3, 4$ [22]. For $N > 4$ there is no limit because the SN temperature is not high enough to produce the heavier KK gravitons efficiently.

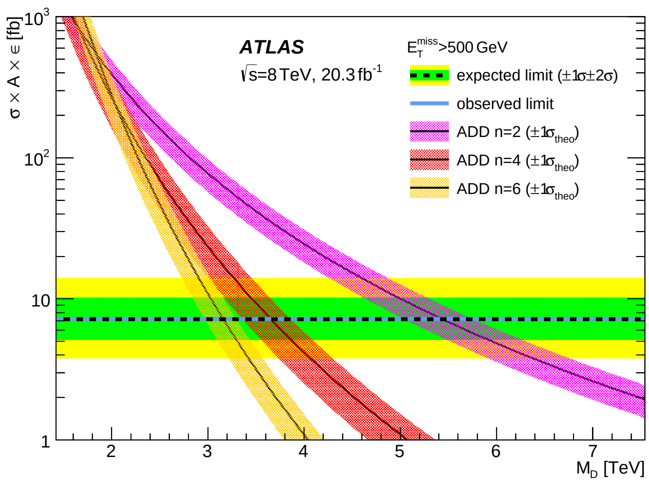

Figure 1: Limits from the ATLAS LHC experiment on LED models with $N = 2, 4, 6$ from a search for a jet plus missing energy using $20\,\mathrm{fb}^{-1}$ of data at $\sqrt{8}$ TeV [21].

2 Warped Extra Dimensions

A second approach to diluting the effective strength of gravity makes use of localization within an extra dimension. This occurs in theories with a *warped* extra dimension in which the full five-dimensional spacetime has a net curvature but the four-dimensional geometry at large distances is effectively flat. The standard example is the Randall-Sundrum (RS) model in which there is a single extra dimension of finite size $w \in [0, \pi\, r_c]$ with metric [8, 9]

$$ds^2 = e^{-2kw}\eta_{\mu\nu}dx^\mu dx^\nu - dw^2 \ , \tag{27}$$

where k corresponds to the five-dimensional curvature. We illustrate the warped geometry of the RS background in Fig. 2.

The RS spacetime is a solution of Einstein's equations in $d = 5$ with a bulk cosmological constant [8, 9]. A net zero cosmological constant can be obtained in the $d = 4$ low-energy effective theory by tuning the tensions of the branes making up the boundaries at $w = 0,\ \pi r_c$. The metric of Eq. (27) is said to be *warped*, the boundary at $w = 0$ is called the *UV brane*, the boundary at $w = \pi r_c$ is called the *IR brane*, and the space in between is called the *bulk*. Similar warped configurations have been found in string theory [13, 14], and the RS configuration can be thought of as an effective theory for them. Note that for this classical solution to be trustworthy, the curvature must be somewhat smaller than the fundamental scale of gravity, $k/M_* \lesssim 1$ [8].

Warping addresses the electroweak hierarchy problem by scaling down the fundamental input parameter $M_* \sim M_{\mathrm{Pl}}$ to exponentially lower values as one moves from the UV brane to the IR brane. In particular, we will see that if the Higgs field is localized on the IR brane, the Higgs mass parameter is warped down to near the weak scale for $kr_c \sim 11$. Related

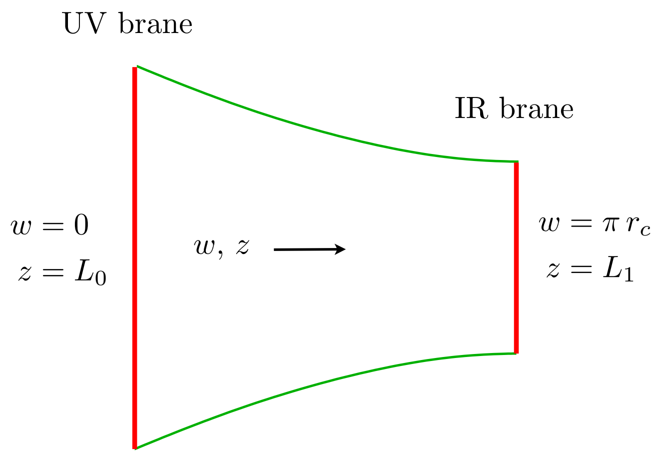

Figure 2: Simple picture of the RS spacetime.

to this, the mass splittings of the KK modes of fields propagating in the bulk end up near the weak scale as well. This implies that some of the SM fields could live in the full five dimensions without contradicting existing experimental limits, in contrast to LED scenarios. By placing the SM fermions in the bulk and localizing them in different regions, we will also see that the large spread in SM fermion masses can be explained in a simple way.

2.1 Warping and the Planck Scale

Let us begin by examining the effects of warping on dimensionful parameters in the theory. The Einstein-Hilbert action describing gravity in the theory is

$$S_{grav} = M_*^3 \int d^4x \int_0^{\pi r_c} dw \, \sqrt{G}\, R^{(5)} \ , \tag{28}$$

where M_* is the fundamental Planck scale, G_{MN} is the $d = 5$ metric, and $R^{(5)}$ is the Ricci scalar built from it. Consider fluctuations of the $d = 4$ components of the metric around the RS background,

$$G_{MN} = \begin{pmatrix} G_{\mu\nu}(x, w) & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{29}$$

where

$$G_{\mu\nu} = e^{-2kw} \overline{g}_{\mu\nu}(x) \ . \tag{30}$$

This is just a w-dependent rescaling of a $d = 4$ metric, and implies that

$$\sqrt{G} R^{(5)} = \sqrt{G}\, R^{(4)} = \sqrt{-\overline{g}}\, \overline{R}^{(4)}\, e^{-2kw} \ , \tag{31}$$

where $\overline{R}^{(4)}$ is the Ricci scalar built out of $\overline{g}_{\mu\nu}$. Putting this result back into the original Einstein-Hilbert action and noting that all the w dependence of the integrand lies in the exponential factor, we find

$$S_{grav} \;\to\; M_*^3 \int d^4x \int_0^{\pi r_c} dw \, \sqrt{-\overline{g}} \, \overline{R}^{(4)} \, e^{-2kw} \tag{32}$$

$$= \frac{M_*^3}{2k} \left(1 - e^{-2\pi k r_c}\right) \int d^4x \, \sqrt{-\overline{g}} \, \overline{R}^{(4)} \; . \tag{33}$$

Comparing to the standard Einstein-Hilbert action, we can identify the $d = 4$ Planck scale we see as

$$M_{\rm Pl}^2 = \frac{M_*^3}{k} \left(1 - e^{-2\pi k r_c}\right) \; . \tag{34}$$

For moderate values of kr_c and $M_* \sim k$, this gives an effective $d = 4$ Planck scale of about the same order as the $d = 5$ Planck scale M_* and the curvature k. Thus, there is no strong volume dilution as in LED.

2.2 Warping and the Higgs Scale

To see how the RS spacetime helps with the hierarchy problem, consider next the action for a Higgs field H confined to the IR brane at $w = \pi r_c$:

$$S_{Higgs} = \int d^4x \int_0^{\pi r_c} dw \, \sqrt{G} \left[G^{\mu\nu} \partial_\mu H^\dagger \partial_\nu H - V(|H|)\right] \delta(w - \pi r_c) \tag{35}$$

$$= \int d^4x \, e^{-4\pi k r_c} \left[e^{2\pi k r_c} \eta^{\mu\nu} \, \partial_\mu H^\dagger \partial_\nu H - V(|H|)\right] \tag{36}$$

$$= \int d^4x \, \left[\eta^{\mu\nu} \, \partial_\mu \widetilde{H}^\dagger \partial_\nu \widetilde{H} - e^{-4\pi k r_c} V(|e^{\pi k r_c} \widetilde{H}|)\right] \; , \tag{37}$$

where in the last line we have changed variables to $H = e^{\pi k r_c} \widetilde{H}$ to make the kinetic term canonical. If the original Higgs potential for H takes the standard form, $V(|H|) = mu^2 |H|^2 + \lambda |H|^4/2$, the rescaled potential in terms of the new variables is

$$e^{-4\pi k r_c} V(|e^{\pi k r_c} H|) \;=\; -\left(\mu^2 e^{-2\pi k r_c}\right) |\widetilde{H}|^2 + \frac{\lambda}{2} |\widetilde{H}|^4 \; . \tag{38}$$

The key feature here is that the dimensionful Higgs mass parameter μ^2 has been warped by the factor $e^{-2\pi k r_c}$.

This warping can be used to address the electroweak hierarchy problem. Naturalness suggests $\mu^2 \sim M_*^2 \sim M_{\rm Pl}^2$. Thanks to the warp factor, the natural value of the effective Higgs mass parameter on the IR brane can be near the weak scale provided the warp factor is big enough, which corresponds numerically to $kr_c \sim 11$:

$$\mu_{eff} \;\sim\; M_* e^{-\pi k r_c} \;\sim\; \text{TeV} \qquad (kr_c \sim 11) \; . \tag{39}$$

We will show below how such a mildly large radius of the extra dimension can come about.

More generally, it is not hard to check that any dimensionful quantity on the IR brane is warped by the same amount. In particular, higher-dimensional operators suppressed by the fundamental Planck scale in the bulk become much less suppressed near the IR brane. For example, a four-fermion operator $\mathcal{O}(w) = (\overline{\psi}\psi)^2/M_*^2$ defined in the bulk becomes (after rescaling to get a canonical kinetic normalization at $w = \pi r_c$)

$$\mathcal{O}(w \to r_c) \to \frac{1}{(e^{-\pi k r_c} M_*)^2}\,(\overline{\psi}\psi)^2 \ . \tag{40}$$

This additional feature presents a new challenge: it implies that the range of validity of the effective theory near $w = \pi r_c$ is limited to energies below about $4\pi\, e^{-\pi k r_c} M_*$. More generally, we will argue below that the local cutoff for physics localized near a given value of w is on the order of $\Lambda_{UV} \sim e^{-\pi k w} M_*$.

2.3 Boson KK Modes in RS

Fields that propagate in the warped extra dimension can be expanded in a tower of KK modes, much like we did in the flat background of LED. The new feature that arises in the warped background is that the KK wavefunctions in the extra dimension take on a more complicated form and tend to be localized within the extra dimension.

When discussing KK modes, it will be convenient to use a different coordinate for the extra dimension than in Eq. (27). Let us define

$$z = (1/k)e^{kw} \ , \qquad z \in [L_0,\, L_1] \ , \tag{41}$$

with $L_0 = 1/k$ and $L_1 = e^{\pi k r_c} L_0$. In terms of the new variable, we have

$$ds^2 = \left(\frac{L_0}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right) = \left(\frac{L_0}{z}\right)^2 \eta_{MN}\, dx^M dx^N \ . \tag{42}$$

All coordinates are now treated similarly, and this form is said to be *conformally flat*.

2.3.1 Vector KK Modes

It will be convenient to begin by examining the KK expansion of a $U(1)$ gauge field in RS. The basic action is

$$S = -\frac{1}{4}\int d^4x \int_{L_0}^{L_1} dz\, \sqrt{G}\, G^{MA} G^{NB} F_{MN} F_{AB} \ , \tag{43}$$

where $F_{MN} = (\partial_M A_N - \partial_N A_M)$.[3] We will also assume Neumann BCs,

$$\partial_Z A_M|_{z=L_0,L_1} = 0 \ . \tag{44}$$

[3]Note that since this combination is antisymmetric in spacetime indices, we can replace gravity-covariant derivatives with regular derivates.

This theory has a gauge invariance under

$$A_M \;\to\; A_M - \frac{1}{g}\,\partial_M\alpha\ , \tag{45}$$

for any nice function $\alpha(x,z)$. We can use this invariance to choose a gauge such that $A_5 = 0$ everywhere. As a result, $F_{\mu z} = -\partial_z A_\mu$ and $F_{\mu\nu}$ are the only non-zero components of the field strength. Applying this to the action together with $G_{MN} = (L_0/z)^2\eta_{MN}$, $G^{MN} = (z/L_0)^2\eta^{MN}$, $\sqrt{G} = (L_0/z)^5$, and integrating by parts, we find

$$\begin{aligned} S \;=\; &\int d^4x \int_{L_0}^{L_1} dz\,\left(\frac{L_0}{z}\right)\frac{1}{2}\left(A_\alpha\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)A_\beta\,\eta^{\alpha\mu}\eta^{\beta\nu}\right. \\ &\left. -\left(\frac{z}{L_0}\right)A_\mu\,\partial_z\left[\left(\frac{L_0}{z}\right)\partial_z\right]A_\nu\,\eta^{\mu\nu}\right) \end{aligned} \tag{46}$$

The first term is the usual $d = 4$ kinetic piece while the second is new.

To make further progress, let us expand the vector field $A_\mu(x,z)$ in a terms of a set of orthonormal basis functions $\{f_n\}$ on the fifth dimension,

$$A_\mu(x,z) = \sum_n A_\mu^{(n)}(x)\,f_n(z)\ , \tag{47}$$

with orthonormality defined by the relation

$$\int_{L_0}^{L_1} dz\,\left(\frac{L_0}{z}\right)\,f_m(z)f_n(z) = \delta_{mn}\ . \tag{48}$$

A particularly nice choice of basis functions are the eigenfunctions of the differential operator

$$D = \left(\frac{z}{L_0}\right)\partial_z\left[\left(\frac{L_0}{z}\right)\partial_z\right]\ . \tag{49}$$

By the Sturm-Liouville theorem discussed in Appendix A, the eigenfunctions of this operator form a complete and orthonormalizable set with respect to the inner product defined above. In particular, with $Df_n = -m_n^2 f_n$, the action becomes

$$S = \frac{1}{2}\int d^4x\ \sum_n A_\alpha^{(n)}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + m_n^2\right)A_\beta^{(n)}\,\eta^{\alpha\mu}\eta^{\beta\nu} \tag{50}$$

This is just a KK tower of massive $d = 4$ vectors with masses m_n.

To determine the basis functions, we must solve $Df_n = -m_n^2 f_n$ subject to the Neumann boundary conditions. For $m_n^2 = 0$, the only solution is

$$f_0(z) = 1/\sqrt{L_0\ln(L_1/L_0)}\ . \tag{51}$$

This is just a constant profile in the extra dimension, and we call it the *zero mode*. For $m_n^2 > 0$, the solutions are related to Bessel functions

$$f_n(z) = N_n\left(\frac{z}{L_0}\right)\left[J_1(m_n z) + \beta_n Y_1(m_n z)\right]\ . \tag{52}$$

Applying the Neumann BC at $z = L_0$ gives

$$\beta_n = -\frac{J_0(m_n L_0)}{Y_0(m_n L_0)} = \frac{\pi}{2} \Big/ \left[\gamma - \ln(m_n L_0/2)\right] \ , \tag{53}$$

while at $z = L_1$ we must have

$$\beta_n = -\frac{J_0(m_n L_1)}{Y_0(m_n L_1)} \ . \tag{54}$$

To satisfy Eqs. (53,54) simultaneously, the masses m_n can take only certain discrete values. For $n \gg 1$ and $L_0 \ll L_1$, the approximate solutions are

$$m_n \simeq \frac{\pi}{L_1}(n - 1/4) \tag{55}$$

$$N_n \simeq \frac{1}{\sqrt{n - 1/4}} m_n \sqrt{L_0} \ , \tag{56}$$

where the N_n factors are fixed by the normalization condition on f_n.

In realistic RS scenarios, we expect $L_0^{-1} = k \sim M_{\rm Pl}$ and $L_1^{-1} = k e^{-\pi k r_c} \sim$ TeV. Our results show that the spectrum of states in the $d = 4$ effective theory consist of a massless vector with constant profile together with a tower of massive KK modes with masses near the TeV scale, $m_n \sim n L_1^{-1} \sim n \times$ TeV. Looking at the KK mode profiles based on the asymptotic forms of Bessel functions listed in Appendix B, they go like

$$f_n(z) \sim \begin{cases} \sqrt{z} m_n \, (\textit{oscillation}) & ; \quad z \sim L_1 \\ -\sqrt{L_0}\, m_n / \ln(m_n L_0) & ; \quad z \ll L_1 \end{cases} \ , \tag{57}$$

where (*oscillation*) refers to a sine or cosine factor. Comparing these limits, we see that $f_n(z)$ has a large amplitude near $z \sim L_1$ but approaches a small constant for $z \ll L_1$. In this sense, the vector KK modes are *localized* near the IR brane.

Finally, let us make two additional comments. First, with Dirichlet boundary conditions the zero mode disappears entirely, but the KK modes are similar. And second, our gauge choice $A_5 = 0$ is related to the spectrum. The zero mode has two degrees of freedom while the massive KK modes all have three. The extra massive degrees of freedom can be understood as coming from the additional gauge component of A_M.

2.3.2 Graviton KK Modes

A similar analysis can be applied to the graviton field. The counting of components is very similar to the LED scenario discussed previously, and we will concentrate exclusively on the $h_{\mu\nu}(x, z)$ components for now with the gauge choices $\partial^\mu h_{\mu\nu} = 0 = h^\mu{}_\mu$. As for the vector, we can expand the field in terms of a complete basis,

$$h_{\mu\nu}(x, z) = \sum_n g_n(z) h^{(n)}_{\mu\nu}(x) \ , \tag{58}$$

with orthonormality defined through

$$\int_{L_0}^{L_1} dz \,\left(\frac{L_0}{z}\right)^3 g_m\, g_n = \delta_{mn} \ . \tag{59}$$

These basis functions can be chosen to satisfy

$$\left(\frac{z}{L_0}\right)^3 \partial_z \left[\left(\frac{L_0}{z}\right)^3 \partial_z\right] g_n = -m_n^2 g_n \ . \tag{60}$$

With these choices, the quadratic part of the $d = 5$ Einstein-Hilbert action separates nicely into a sum over independent KK modes.

To reproduce standard gravity in the $d = 4$ effective theory at low energies, there should be a massless zero mode to identify with the standard graviton. Such a mode emerges if the $d = 5$ field satisfies Neumann BCs on both branes, and we will assume this to be the case. The zero mode profile is then constant with

$$g_0(z) \ \simeq\ \sqrt{2/L_0} \ . \tag{61}$$

The higher KK modes have profiles

$$g_n(z) = N_n \left(\frac{z}{L_0}\right)^2 [J_2(m_n z) + \beta_n Y_2(m_n z)] \ . \tag{62}$$

Applying the Neumann BCs at $z = L_0,\ L_1$ then fixes β_n and selects the allowed discrete KK masses $m_n \simeq (\pi/L_1)(n+1/4)$. Just like for the vector, these are on the order of $m_n \sim nL_1^{-1}$.

It is instructive to compare the KK mode profiles of the vector and graviton fields (assuming Neumann BCs for both). Note that they are defined with respect to different weight functions in the orthogonality conditions of Eqs. (48,59): (L_0/z) for the vector, and $(L_0/z)^3$ for the graviton. To account for this, we should compare the rescaled profiles $f_n(z)$ and $(L_0/z)g_n(z)$.[4] After the rescaling, we see that the vector and graviton KK profiles are parametrically similar and are localized towards the IR brane at $z = L_1$. In contrast, the rescaled graviton zero mode goes like $\sqrt{L_0}/z$ and is largest near the UV brane $z = L_0$, while the vector zero mode is flat. This localization of the graviton zero mode in UV is closely related to apparent weakness of gravity relative to the local scale $(L_0/z)M_*$ for an observer at coordinate value z in the extra dimension.

2.3.3 Scalar KK Modes

...

[4] Note that $(L_0/z)dz = dw$, so this is the natural weight to use to.

2.4 Fermion KK Modes in RS

Fermions are bit more complicated than bosons in the RS background. In $d = 5$ flat spacetime, the set of gamma matrices corresponding to the minimal fermionic representation of Lorentz are

$$\Gamma^A = (\gamma^\mu, i\gamma^5) \ , \tag{63}$$

which clearly satisfy $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. In particular, we see that from the $d = 4$ point of view the representations are not chiral, and the minimal $d = 5$ fermion representation coincides with a Dirac fermion in $d = 4$. Some extra work will therefore be needed to get chiral fermions in the low-energy effective theory.

A second challenge in RS is that these fermions are defined with respect to a flat background, but the RS background is warped. To accommodate this, the standard procedure is to define a locally flat *tangent space* at any point in spacetime, and build fermion representations within each tangent space. To connect between tangent spaces defined at different points, it is natural to introduce the *vielbein* $e_M^A(x)$ that relates spacetime and tangent space indices via

$$G^{MN} e_M^A e_N^B = \eta^{AB} \ . \tag{64}$$

With the vielbein, it is possible to build a *spin connection* that permits the construction of a gravity-covariant derivative D_M on objects defined on the tangent space. We will not go into the details of this process for RS, but just quote the final result expressed in terms of spacetime indices. A nice introductory discussion in the context of RS spacetimes is given in Ref. [23].

After including and simplying the effects of the warped RS background, the basic action for a fermion Ψ propagating in the full $d = 5$ spacetime is [23]

$$\begin{aligned} S &= \int d^4x \int_{L_0}^{L_1} dz\ \left(\frac{L_0}{z}\right)^4 \left[\frac{i}{2}\,\overline{\Psi}\,\Gamma^M \overleftrightarrow{\partial}_M \Psi - \left(\frac{L_0}{z}\right) m\,\overline{\Psi}\Psi\right] &(65)\\ &= \int d^4x \int_{L_0}^{L_1} dz\ \left(\frac{L_0}{z}\right)^4 \overline{\Psi} \begin{pmatrix} -\partial_z + (2-c)/z & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & \partial_z - (2+c)/z \end{pmatrix} \Psi &(66)\\ &\quad + \int d^4x\ \left[\left(\frac{L_0}{z}\right)^4 \left(\bar{\Psi}_L\Psi_R - \bar{\Psi}_R\Psi_L\right)\right]_{L_0}^{L_1} . \end{aligned}$$

In the second line we have integrated by parts and defined

$$m = c/L_0 \ , \tag{67}$$

so that c is the fermion mass in units of the fundamental scale L_0^{-1}.

We would like to find a KK decomposition for Ψ similar to what we had for bosonic fields. For this, we will choose BCs such that the boundary term in Eq. (66) vanishes and look for expansions of the form

$$\Psi_L(x,z) = \sum_n f_n^L(z)\psi_L(x) \ , \qquad \Psi_R(x,z) = \sum_n f_n^R(z)\psi_R(z) \ . \tag{68}$$

These expansions will lead to a diagonal form in Eq. (66) if [16]

$$\left(\partial_z - \frac{2-c}{z}\right) f_n^L \;=\; -m_n f_n^R \tag{69}$$

$$\left(\partial_z - \frac{2+c}{z}\right) f_n^R \;=\; m_n f_n^L \;. \tag{70}$$

Plugging one equation into the other separates f_n^L and f_n^R and gives

$$\left(\frac{z}{L_0}\right)^4 \partial_z \left[\left(\frac{L_0}{z}\right)^4 \partial_z\right] f_n^L \;=\; -\left(m_n^2 - \frac{c^2+c-6}{z^2}\right) \tag{71}$$

$$\left(\frac{z}{L_0}\right)^4 \partial_z \left[\left(\frac{L_0}{z}\right)^4 \partial_z\right] f_n^R \;=\; -\left(m_n^2 - \frac{c^2-c-6}{z^2}\right) \;. \tag{72}$$

Both of these equations are of the Sturm-Liouville form (for appropriate BCs), and their solutions will form a complete set eigenfunctions that are orthonormalizable in the sense

$$\int_{L_o}^{L^1} dz\, \left(\frac{L_0}{z}\right)^4 f_m^L f_n^L \;=\; \delta_{mn} \;=\; \int_{L_o}^{L^1} dz\, \left(\frac{L_0}{z}\right)^4 f_m^R f_n^R \;. \tag{73}$$

The form of these solutions, however, will depend on the BCs we apply.

For $m_n = 0$ the left- and right-handed modes decouple in Eqs. (69,70) and we find the *zero mode* solutions [16]

$$f_0^L \;=\; A_0 \left(\frac{z}{L_0}\right)^{2-c} \tag{74}$$

$$f_0^R \;=\; C_0 \left(\frac{z}{L_0}\right)^{2+c} \;. \tag{75}$$

These solutions must be augmented by BCs on $\Psi(x,z)$ at $z = L_0,\, L_1$. For the choice $\Psi_R|_{L_0,L_1} = 0$, the boundary term in Eq. (66) vanishes and we are also forced to set $C_0 = 0$. This is a useful result: this choice removes the right-handed zero mode but does not impose a condition on the left-handed zero mode. Thus, we obtain a chiral spectrum of zero modes, which is precisely what we want to describe the fermions of the SM. An alternate choice is $\Psi_L|_{L_0,L_1} = 0$, which now projects out the left-handed zero mode by forcing $A_0 = 0$.

Turning next to solutions with non-zero m_n from Eqs. (71,72), one obtains [16]

$$f_n^L(z) \;=\; \left(\frac{z}{L_0}\right)^{5/2} \left[A_n J_{c+1/2}(m_n z) + B_n Y_{c+1/2}(m_n z)\right] \tag{76}$$

$$f_n^R(z) \;=\; \left(\frac{z}{L_0}\right)^{5/2} \left[C_n J_{c-1/2}(m_n z) + D_n Y_{c-1/2}(m_n z)\right] \tag{77}$$

for some constants A_n, B_n, C_n, D_n. To match up with the first-order forms of Eqs. (69,70), these constants must be related by $A_n = C_n$ and $B_n = D_n$. The two remaining constants are then fixed by the BCs. For example, $\Psi_R|_{L_0,L_1} = 0$ gives two conditions on C_n and D_n that can be only be solved non-trivially for certain discrete values of the KK masses $m_n \sim \pi\, n/L_1$. This fixes everything up to the overall normalization.

As we did for the graviton expansion, it is instructive to look at how the various fermion wavefunctions are distributed across the extra dimension. Recall that $dw = (L_0/z)dz$ is the natural measure to compare to, while the fermion KK wavefunctions are orthonormal with respect to the measure $(L_0/z)^4 dz$. To account for this, we should really examine $(L_0/z)^{3/2} f_n^{L,R}(z)$. After rescaling, the fermion KK modes are localized towards the IR brane ($z = L_1$), while for the zero modes we have

$$\left(\frac{L_0}{z}\right)^{3/2} f_0^L(z) \;=\; A_0 \left(\frac{z}{L_0}\right)^{1/2-c} \tag{78}$$

$$\left(\frac{L_0}{z}\right)^{3/2} f_0^R(z) \;=\; C_0 \left(\frac{z}{L_0}\right)^{1/2+c} . \tag{79}$$

Thus, a left-handed zero mode is localized near the IR for $c < 1/2$, while a right-handed zero mode is localized in the IR for $c > -1/2$. We will see an interesting physical consequence of these features in the next subsection.

2.5 RS with the SM on the IR Brane (RS1)

The original SM embedding in RS (by RS) has the entire SM confined to the IR brane at $z = L_1$ ($w = \pi\, r_c$) [8]. It is sometimes called RS1. Only gravity propagates in the bulk, and the only KK modes are those of the graviton $h_{MN}(x, z)$. The coupling of the SM to gravity takes the standard form up to localization on the brane,

$$S_{RS1} \;=\; \int d^4x \int_{L_0}^{L_1} dz\, \sqrt{G}\, \mathscr{L}_{SM}(G)\, \delta(z - L_1) \tag{80}$$

$$\supset\; -\int d^4x\, \frac{1}{M_{\rm Pl}}\, T^{\mu\nu}_{SM} \left[h^{(0)}_{\mu\nu} + \frac{L_1}{L_0} \sum_{n=1}^{\infty} h^{(n)}_{\mu\nu} \right] , \tag{81}$$

where $T^{\mu\nu}_{SM}$ is the usual energy-momentum tensor of the SM. As expected, the graviton zero mode couples to the SM in usual way with strength $1/M_{\rm Pl}$. However, the important new feature is that the higher graviton KK modes have a coupling to the SM that is enhanced by a large relative factor of $L_1/L_0 = e^{\pi\, k\, r_c}$, corresponding to $(L_1/L_0)/M_{\rm Pl} \sim 1/\,\text{TeV}$. Again, this can be understood based on the KK wavefunctions, with the zero mode localized near the UV brane and the KK modes localized near the IR brane.

The new signals predicted by this theory are KK graviton excitations. With the Higgs on the IR brane, recall that the natural value of its mass is on the order of $\mu_{eff} \simeq M_* e^{-\pi\, k\, r_c} = (M_{\rm Pl}^2/L_0)^{1/3}(L_0/L_1) \sim (1/L_1)$, while the masses of the KK gravitons are close to $m_n \simeq \pi\, n/L_1$. Thus, we expect $(1/L_1) \sim$ TeV by naturalness with higher values requiring a fine-tuning of the Higgs mass parameter.

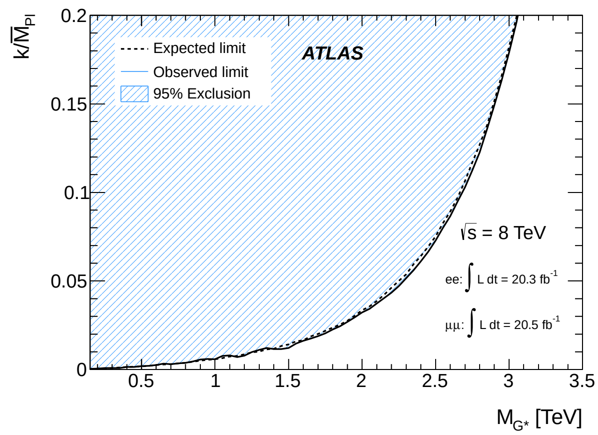

Figure 3: Excluded region of the RS1 parameter space from ATLAS dilepton searches with $20\,\text{fb}^{-1}$ of data at $\sqrt{s} = 8$ TeV. Figure from Ref. [25].

Graviton KK modes can be created in pp collisions at the LHC and detected efficiently when they decay to leptons. One of their characteristic signals is a resonance in the dilepton invariant mass spectrum [24]. In Fig. 3 we show the exclusions on the KK gravitons in RS1 derived by the ATLAS collaboration using $20\,\text{fb}^{-1}$ of data at $\sqrt{s} = 8\,\text{TeV}$ [25]. The excluded region is shown as a function of the lightest KK graviton mass $m_1 = M_{G^*}$ and $k/M_{\text{Pl}} = (L_1/L_0)/M_{\text{Pl}}$. Note that this second combination of parameters how strongly the KK gravitons couple to the SM (Eq. (81)). Unless k/M_{Pl} is particularly small, the bounds on m_1 are the order of a few TeV.

2.6 RS with the SM in the Bulk

A second way to put the SM in the RS background is to allow all SM fields except for the Higgs to propagate in full five dimensions. The Higgs field must remain on the IR brane (or very close to it) for its mass parameter to be naturally small. An important feature of this extension of the SM is that it provides a nice explanation for the broad range of Yukawa couplings in the SM based on localization in the extra dimension.

Let us begin with the SM fermions. While fermion representations in $d = 5$ are non-chiral, we saw above that either the left- or right-handed zero mode of a bulk fermion can is projected out depending on its boundary conditions. To see how this can be applied to the SM, let us define $\Psi_Q(x,z) = (\Psi_{Q_L}, \Psi_{Q_R})^t$ to be a bulk fermion with gauge quantum numbers $(3, 2, 1/6)$. Note that $SU(2)_L$ acts on both the left- and right-handed components of Ψ_Q. By choosing BCs with $\Psi_{Q_R}|_{L_0,L_1} = 0$, only the left-handed component has a zero mode with profile

$$\left(\frac{L_0}{z}\right)^{3/2} f_0^{Q_L}(z) \;\propto\; \left(\frac{z}{L_0}\right)^{1/2-c_Q} , \tag{82}$$

where c_Q is the bulk mass parameter for Ψ_Q. We identify this zero mode with the left-handed SM quark doublet Q. The KK modes of Ψ_Q have profiles as in Eqs. (76,77) with $c \to c_Q$, and include both left- and right-handed states at each level. In the same way, we can also introduce $\Psi_U(x,z) = (\Psi_{U_L}, \Psi_{U_R})^t = (3,1,2/3)$ and choose BCs such that only the right-handed component has a zero mode with profile

$$\left(\frac{L_0}{z}\right)^{3/2} f_0^{U_R}(z) \;\propto\; \left(\frac{z}{L_0}\right)^{1/2+c_U} . \tag{83}$$

As before, the KK modes have both left- and right-handed components.

This procedure can be extended to all the fermions of the SM, with one bulk fermion introduced for each SM state and three generations in all. At this point, the fermion zero modes are massless. They can get masses by coupling to the Higgs on the IR brane in the form (for the up-type quarks)

$$\begin{aligned} S \quad &\supset \quad -\int d^4x \int_{L_0}^{L^1} dz\; \left(\frac{L_0}{z}\right)^5 y\, \overline{\Psi}_Q \Psi_U\, H\, \delta(z-L_1) + (h.c.) && (84)\\ &\supset \quad -\int d^4x\; y\, A_0^Q C_0^U \left(\frac{L_1}{L_0}\right)^{1/2-c_Q} \left(\frac{L_1}{L_0}\right)^{1/2-c_U} \overline{Q}_L u_R\, H + (h.c.)\;, && (85)\end{aligned}$$

where we have identified the zero modes of Ψ_Q and Ψ_R with the corresponding SM fermions Q and u_R, and A_0^Q and A_0^U are normalization factors for the zero-mode wavefunctions. The form of Eq. (85) is precisely that of the up-quark Yukawa coupling to the SM Higgs with the effective strength

$$y_u^{eff} = y\, A_0^Q C_0^U \left(\frac{L_1}{L_0}\right)^{1/2-c_Q} \left(\frac{L_1}{L_0}\right)^{1/2-c_U} . \tag{86}$$

Its value depends on how the zero modes are localized in the bulk, with IR-localized modes giving more overlap with the Higgs and larger Yukawa couplings. For variations in the c_i coefficients by an amount of order unity with $y \sim 1$, very large differences in the effective Yukawa couplings can arise. For example, the left- and right-handed top quarks should be localized near the IR brane, while the first and second generations should lie more towards the UV brane. Note that the c_i coefficients will be different within each generation, and this can also allow for a natural origin of the hierarchical CKM matrix.

Turning next to the SM vector bosons, recall that we found above that their zero modes have flat profiles in the extra dimension (assuming Neumann BCs), Eq. (51). This is an important result because it implies that the gluon and the photon will have the same couplings to all zero-mode and KK fermions regardless of how they are localized. The same applies for the graviton as well. In contrast, the vector KK modes are localized mostly near the IR brane, and they will couple more strongly to the fermions of the third generation and to other KK states than to the light generations.

Given these localization arguments, a prominant LHC signal of RS theories with the SM in the bulk is the production of KK gluons [26, 27]. These have strong production rates, and they tend to decay predominantly into top-antitop pairs. In Fig. 4, we show the limits on

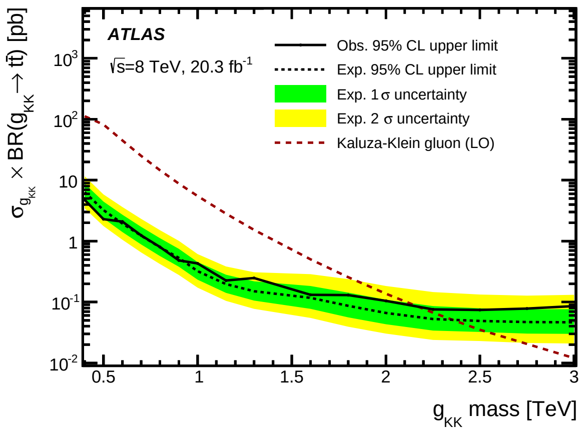

Figure 4: Exclusions on the bulk RS KK gluon mass from ATLAS $t\bar{t}$ searches with $20\,\text{fb}^{-1}$ of data at $\sqrt{s} = 8$ TeV. Figure from Ref. [28].

KK gluons with $k/M_{\text{Pl}} = 1$ obtained by the LHC ATLAS experiment with $20\,\text{fb}^{-1}$ of data at $\sqrt{s} = 8$ TeV in the $t\bar{t}$ channel [28]. The limits extends up to $m_{KK} > 2$ TeV, which is much larger than twice the top mass. An interesting technical challenge in these searches is that that tops emitted in the decay are expected to be highly boosted, implying that the subsequent top quark decay products (from the dominant $t \to W^+b$ decay) will be very collimated and the usual top identification methods are no longer efficient. To distinguish such boosted tops from regular QCD jets more effectively, a set of techniques known at *top tagging* (or *jet substructure*) can be applied [29, 30].

Bulk RS models also face a number of indirect constraints that typically require addtional structure if they are to be avoided while maintaining $(1/L_1) \sim$ TeV. Precision measurements of the SM weak vector bosons find that they interact with fermions (and the Higgs) as predicted by the SM to a very high level of accuray. Thus, any deviation in their couplings must be very small. In bulk RS, the Higgs VEV deforms the $W^\pm$ and Z^0 wavefunctions from their would-be flat zero-mode profiles and modifying their couplings to the SM fermions, and a further shift is generated by the four-fermion operators generated by integrating out the massive KK modes [31]. To repair this, an additional *custodial* bulk gauge symmetry (and associated bulk gauge bosons) is usually introduced together with new bulk fermions connecting mainly to the third generation [32, 33]. New mixing of quark flavours can also arise, putting even more constraints on the theory [34, 35].

2.7 Radion Stabilization

An important issue that we have not yet addressed is the stability of the five-dimensional RS spacetime. Without any additional structure, the separation of the UV and IR branes is classically metastable with respect to small variations. This can be seen by examining fluctuations in the G_{55} component of the metric, whose zero mode we identify as a $d = 4$

Lorentz scalar *radion* field. Fortunately, it is relatively straightforward to stabilize the radion such that $kr_c \sim 11$ as required for RS to account for the large ratio between the observed Planck and weak scales [36].

The generic model of radion stabilization is the Goldberger-Wise (GW) mechanism of Refs. [36, 37]. A massive bulk scalar field is added to the theory with couplings on the branes such that it is forced to get different VEVs at each brane. This forces the zero mode of the scalar field to have a non-trivial profile in the bulk. Studying how the energy of the configuration varies with the brane separation, it is found to be minimum at a non-zero value that balances the gradient energy (which is lower for larger brane separations) with the mass-term energy (which is lower for smaller brane separations). With some mild adjustment of the parameters of the bulk scalar, a spacetime with $kr_c \sim 11$ can be achieved.

In addition to stabilizing the radius of the fifth dimenison, the GW scalar also generates a mass for the radion field [36, 37, 38]. Recall that we first encountered the radion in the context of LED in Sec. 1.2. A similar analysis applies to the additional fluctuations of the metric G_{MN} around the RS background, with the physically modes again being the graviton and the radion (and their KK modes). Without radius stabilization, the radion field in RS (and LED) is massless. Applying the GW mechanism, there is a modest backreaction on the spacetime that generates a mass for the radion [15, 36, 39]. The coupling of the radion to SM matter is similar to that of the Higgs boson, and radion-Higgs mixing can occur as well [39].

3 Warping and Strong Coupling

A deep relation is thought to exist between warped theories with gravity in $d+1$ dimensions and certain classes gauge theories without gravity in d dimensions. This relation is usually called the *AdS/CFT correspondence*, and is often referred to as *holography*. It was first proposed in certain specific string theory compactifications [40], but is thought to hold much more generally [41, 42, 43]. In particular, the correspondence suggests that RS theories are an alternate description of technicolor-like theories with the SM Higgs and matter fields arising as (partly) composite bound states [44, 45, 46].

3.1 $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM

The AdS/CFT correspondence was first proposed in a specific compactification of type IIB superstring theory [40]. At low energies, the string theory reduces a supergravity effective theory on $AdS_5 \times S^5$, where AdS_5 refers to an infinite $d = 5$ Anti-deSitter space and S^5 is a compact 5-sphere. The key observation of Ref. [40] is that this theory bears an uncanny resemblance to $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) in $d = 4$ (without gravity). Based on this resemblance, Ref. [40] went on the make the bold proposal that the two theories are *dual* to each other at low energy, in that they provide equivalent (but seemingly different) descriptions of the same physics.

Starting with the symmetries of the two systems, the set of isometries of the 5-sphere is $SO(6)$, which is isomorphic to the $SU(4)_R$ global symmetry group of the SYM theory. Furthermore, the set of isometries of AdS_5 coincides with the conformal group in $d = 4$, while the $\mathcal{N} = 4$ SYM theory is known to be a conformal field theory (CFT) with the same spacetime symmetries.

There is also a matching between the free parameters of the two theories. On the supergravity side, the theory is characterized by the AdS_5 curvature radius R_{AdS} and the fundamental string length scale ℓ_s below which stringy excitations start to appear. In the gauge theory, the parameters are the gauge coupling and the number of colours N (gauge group $SU(N)$). Matching between the theories suggests

$$\left(\frac{R_{AdS}}{\ell_s}\right)^4 = 4\pi\, g^2\, N\ . \tag{87}$$

This correspondence is significant because it matches weak coupling on one side to strong coupling on the other. In particular, for the supergravity limit of the string theory to be valid one needs $R_{AdS} \ll \ell_s$. Applying this to the gauge theory via Eq. (87), the supergravity limit corresponds to a large 't Hooft coupling $\lambda = 4\pi\, g^2\, N \gg 1$, implying that the gauge theory is strongly coupled in this limit.

The AdS/CFT correspondence has passed a wide variety of non-trivial checks in this and other theories [43]. It is widely thought to apply very generally, and to represent a deep relation between gravity and gauge theories. The connection to RS theories is that the RS spacetime is a finite slice of AdS_5, bounded by the UV and IR branes, with curvature radius $k^{-1} = R_{AdS}$. While this differs significantly from the original $AdS_5 \times S^5$, there are good reasons to expect that it also corresponds to an approximately conformal gauge theory. We will discuss this in more detail below.

3.2 Conformal Field Theories in $d = 4$

Before discussing RS and its relation to $d = 4$ gauge theories, it will be helpful to cover first a few basic features of conformal field theories (CFTs) in four spacetime dimensions. Conformal invariance is an extension of the Poincaré group (in d dimensions) to include dilatations and special conformal transformations [47]. Dilatations act on spacetime according to

$$x^M \to \lambda\, x^M\ , \tag{88}$$

while special conformal transformations transform it by

$$x^M \to \frac{x^M + x^2\, b^M}{1 + b\cdot x + b^2 x^2}\ . \tag{89}$$

Here, λ and b^M are parameters that define the transformations. Invariance of a QFT under conformal transformations puts very strong restrictions on the operator expectation values in the theory, and is an extremely powerfual calculational tool.

Theories that are invariant under dilatations are said to have *scale invariance*. In $d = 2$ it has been proven that if a theory is scale invariant, it is also invariant under the more general conformal invariance [48]. This has not been proven for $d = 4$, but there are strong arguments that it should be the case [48]. In the discussion to follow, we will assume this to be true and concentrate exclusively on scale transformations. The Noether current for dilatations is

$$D_\mu = x^\alpha T_{\alpha\mu} - K_\mu \ , \tag{90}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the theory and K_μ is some 4-vector with $\partial_\mu K^\mu = 0$. Taking the divergence of D_μ, we see that scale invariance requires

$$\partial_\mu D^\mu = 0 \ = \ T^\mu_{\ \mu} \ . \tag{91}$$

Thus, conformal invariance requires a traceless energy-momentum tensor.

To see how this invariance does and does not work, consider a simple free scalar theory and let us apply a scale transformation to it. We will assume that $x \to x' = e^\alpha x$ and $\phi(x) \to \phi'(x) = e^{-\beta}\phi(e^{-\alpha}x)$ for some parameters α and β. The action transforms according to

$$\begin{aligned} S &= \int d^4x \, \left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{A}{3!}\phi^3 + \frac{\lambda}{4!}\,\phi^4 + \ldots\right] && (92) \\ &\to \int d^4x \; e^{(4\alpha-2\alpha-2\beta)} \left[(\partial\phi)^2 - \frac{1}{2}e^{2\alpha}m^2\phi^2 - \frac{e^\alpha A}{3!}\phi^3 + \frac{\lambda}{4!}\,\phi^4 + \ldots\right] \ . && (93) \end{aligned}$$

Looking at this result, we see that the kinetic term will be invariant for $\beta = \alpha$. Since the kinetic is the minimal term needed to define a theory, this is the natural choice for β. Going next to the other terms, we see that they all pick up factors with powers equal to the mass dimension of their corresponding coupling, and only the quartic term is invariant. The explicit breaking of scale invariance in this theory can be tracked by noting that the dimensionful terms would be invariant if we also had $m^2 \to e^{-2\alpha}m^2$ and $A \to e^{-\alpha}A$.

More generally, it is not hard to show that the tree-level action of a theory is invariant under scale transformations provided it does not have any dimensionful couplings. An important example of this is massless QCD generalized to N colours ($SU(N)$ gauge group) with N_f flavours of massless quarks. The basic action is

$$S = \int d^4x \left(-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} + \sum_{i=1}^{N_f} \bar{q}_i\, i\gamma^\mu D_\mu\, q_i \right) \ , \tag{94}$$

and it is invariant under $x \to e^\alpha x$, $A^a_\mu(x) \to e^{-\alpha}A^a_\mu(e^{-\alpha}x)$, and $q(x) \to e^{-3\alpha/2}q(e^{-\alpha}x)$. Correspondingly, the trace of the energy-momentum tensor built from the tree-level action is zero.

The problem with this picture is that we know that standard QCD (with $N = 3$ and $N_f = 2$ or 3) is not scale invariant, even in the limit of vanishing quark masses. In particular, a physical scale $\Lambda_{QCD} \simeq 200$ MeV emerges where the gauge coupling $\alpha_s = g_s^2/4\pi$ grows strong in its renormalization group (RG) evolution. This physical scale, and the effective scale dependence of α_s, represents an explicit breaking of scale invariance by quantum mechanical effects. Indeed, computing the trace of the energy-momentum tensor gives

$$T^\mu_{\ \mu} \ = \ \partial_\mu D^\mu \ = \ \frac{\beta_s(g_s)}{2g_s^3} G^a_{\mu\nu} G^{a\,\mu\nu} \ , \tag{95}$$

where β_s is the beta function for g_s. At one-loop, it is

$$\beta_s \ = \ \frac{dg_s}{dt} \ = \ -\frac{g_s^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right) + \ldots \tag{96}$$

where $t = \ln(\mu/\mu_0)$ is the RG scale. When β_s is non-zero, we say that scale invariance is *anomalous*, in that it looks like a symmetry of the theory at the level of the tree-level action but is not in the full quantum theory. This is precisely what happens in regular QCD with $N = 3$ and $N_f = 2,\, 3$.

Going beyond regular QCD to other values of N and N_f, an interesting possibility is that $\beta_s \to 0$ at low energies. This occurs in a trivial way for $N_f \gg 11N/2$ and moderate $g_s(\mu_0)$, with $g_s \to 0$ along with $\beta_s \to 0$ as $\mu/\mu_0 \to 0$. At low energies, the theory simply approaches a free theory. An alternative and more interesting behaviour is $\beta_s \to 0$ with a fixed non-zero value of $g_s = g_*$ as $\mu/\mu_0 \to 0$. This is thought to occur in QCD-like theories for certain ranges of values of N and N_f, and the corresponding fixed value of g_* is called a conformal infrared (IR) fixed point. If different initial values of $g_s(\mu)$ flow toward g_* as μ/μ_0 decreases, the fixed point is said to be attractive (or stable) in the IR. Once at g_*, the properties of the theory remain the same going to lower scales. The form of β_s for such a fixed point is shown in Fig. 5. For regular QCD, the corresponding curve would start at the origin and become increasingly negative as g_s increases.

In this picture, the generalized QCD theory is exactly conformal when $g_s = g_*$, and approaches this point in the IR for other values of g_s. Conformal invariance of the theory can therefore arise as an asymptotic limit of the RG flow. Moreover, different theories with the same field content but different values of $g_s(\mu)$ all flow to the same theory in the IR. This property is sometimes called *universality*.

It is also interesting to modify the generalized massless QCD theory with quark mass terms. Suppose we have N and N_f such that the theory has a stable IR fixed point $g_* \ll 1$, and let us add a mass term $m\bar{q}q$ for all flavours. This obviously breaks scale invariance, but it is a numerically small effect for $\mu \gg m$. Starting at some $\mu \gg m$ with $g_s(\mu)$ very close to g_*, the theory will quickly flow to the fixed point. It will remain there as μ is lowered further until $\mu \sim m$. At this point, the quark masses become important, and they can be integrated out of the theory if we are interested in physics at much lower energies. The effective theory for $\mu < m$ now contains only gluons, with the basic gluon kinetic operator of Eq. (94) together with higher-dimensional operators suppressed by powers of m. This theory is not scale invariant, with $\beta_s \simeq -(11N/3)g_s^3/(4\pi)^2$, and flows to larger values of g_s and possibly confines at lower energies.

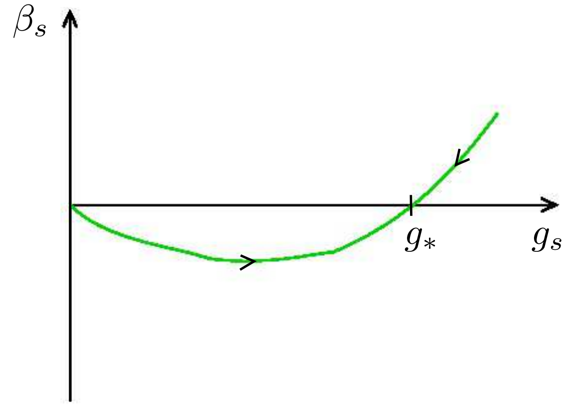

Figure 5: Schematic picture of the RG flow of g_s in generalized massless QCD for values of N and N_f for which there is an IR stable fixed point g_*. The arrows show the direction of evolution of g_s as μ/μ_0 decreases.

A second modification of this story is to keep N colours and N_f massless flavours such that there is a stable IR fixed point at $g_s = g_*$, but now add $\Delta N_f > 11/2$ additional quarks with mass M. For $\mu \gg M$ and perturbative $g_s(\mu)$, the theory flows to lower values of g_s as μ decreases since all quarks (massive and massless) contribute to β_s. When $\mu \sim M$, we can integrate out the heavy quarks to get back N_f massless flavours only, but with new higher-dimensional operators suppressed by M. These operators become increasingly irrelevant for $\mu < M$, and the theory will eventually flow to g_*.

We have only just scratched the surface of the many interesting features of conformal field theories in $d = 4$. Exact conformal invariance itself is only part of the story, and the way in which theories flow into and out of approximately conformal regimes is important too. We have also focussed mainly on perturbative conformal IR fixed points. The fixed coupling g_* can also be large enough that it is no longer a useful pertubative parameter. In this case, the scale dependence of operators can be very different from what would be expected at weak coupling, and the most useful degrees of freedom might no longer be the original set of fields one started with.

3.3 Holography and RS

Generalized AdS/CFT suggests that RS theories are dual to certain types of approximately conformal gauge theories. We will only present here the main features of the connection. More detailed intoductions can be found in Refs. [16, 23, 44, 45, 46].

In this correspondence, moving in the warped extra dimension is related to an RG flow in the gauge theory description. Note that this gauge theory is for a new technicolor-like force that is independent of the SM gauge fields. Increasing $z/L_0 = e^{kw}$ is identified with lowering the RG scale $\mu/M_* \simeq e^{-t}$. Between the branes, $L_0 < z < L_1$, the space is locally AdS_5 and the dual gauge theory is approximately conformal. The UV brane at $z = L_0$ truncates the AdS_5 space and represents an explicit breaking of conformal in the UV, similar to the appearance of heavy quarks with mass M and $\Delta N_f > 11/2$ in the example above. The IR brane at $z = L_1$ also coincides with conformal breaking, but corresponds to a spontaneous breaking in the IR. To see that it is spontaneous rather than explicit, note that the RS theory has a massless radion mode in the absence of a stabilization mechanism corresponding to fluctuations of the IR brane location. This radion matches to the NGB of spontaneous conformal breaking in the gauge theory. To stabilize the radion and give it a mass, the GW mechanism adds a massive bulk scalar. It can be matched to a would-be marginal operator in the gauge theory that grows strong in the IR and adds an explicit breaking of conformal invariance.

These correspondences can also be extended to SM fields in the bulk or on the branes. In general, fields localized in the UV match to fundamental fields in the gauge theory, while fields localized in the IR match to composite operators. The Higgs is the key example of this, but the rest of the SM fields in RS are also expected to be composites in the gauge description when they are localized on the IR brane. More generally, a field that is localized in the bulk is a mixture of fundamental and composite operators.

A Appendix: Sturm-Liouville

A mathematical result that is used all over the place in physics is the Sturm-Liouville theorem. Since we apply it in these notes, we give an informal and simplified overview of it here. The theorem is essentially an application of the orthogonality and completeness of eigenstates of Hermitian operator applied to function spaces.

Let V be the vector space consisting of all reasonably nice functions $\{f(x)\}$ defined on the finite interval $[a, b]$ satisfying the boundary conditions (BCs)

$$f(a) - \xi\, \partial_x f(a) = 0 \ , \qquad f(b) + \xi\, \partial_x f(b) = 0 \ . \tag{97}$$

for some fixed constant ξ. Note that $\xi \to 0$ corresponds to *Dirichlet* BCs and $\xi \to \infty$ to *Neumann* BCs. It is straightforward to check that V is indeed a vector space. What's more, for any nice function $w(x) > 0$ (on $[a, b]$) we can also define an inner product on V with

$$(f,\, g) = \int_a^b dx\; w(x)\, f(x)\, g(x) \ . \tag{98}$$

This is clearly bilinear and positive semi-definite.

Consider now the linear operator on V defined by

$$Df = \frac{1}{w(x)} \left[\partial_x \left(p(x)\partial_x\right) + q(x)\right] f(x) \ . \tag{99}$$

It is not hard to show that this operator is Hermitian on the space of functions with respect to the inner product, namely

$$(Df,\, g) = (f,\, Dg) \ . \tag{100}$$

Proving this just requires a few integrations by parts and application of the boundary conditions satisfied for f and g.

Now let $\{f_n\}$ be the set of eigenvectors of D with eigenvalues λ_n,

$$Df_n = -\lambda_n^2 f_n \ . \tag{101}$$

Just like we use all the time in quantum mechanics, the set of eigenvectors forms a basis for the vector space, and eigenvectors with different eigenvectors are orthonormal (for a nice choice of normalization):

$$(f_m,\, f_n) = \int_a^b dx\; w(x)\, f_m(x)\, f_n(x) = \delta_{mn} \ . \tag{102}$$

In this context, the result is called the *Sturm-Liouville theorem*. We will use this feature to expand fields defined in extra dimensions in a basis of eigenfunctions for the space of functions on the extra dimensions.

B Appendix: Bessel Functions

Bessel functions arise frequently in discussions of warped extra dimensions. We collect here a few results about them that are used in these notes.

Not surprisingly, Bessel functions arise as solutions of the Bessel equation. Specifically, $J_\alpha(x)$ and $Y_\alpha(x)$ are the two independent solutions of the linear ODE

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + (1-\alpha^2/x^2)\right] f(x) = 0 \ . \tag{103}$$

Here, the parameter α is a real number that defined the *order* of Bessel equation (or functions). These solutions have a number of nice properties. Asymptotically, they reduce to

$$J_\alpha(x) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha & ; \quad 0 < x \ll \sqrt{\alpha+1} \\ \\ \sqrt{\frac{2}{\pi x}}\,\cos\left(x - \alpha\pi/2 - \pi/4\right) & ; \quad x \gg |\alpha^2 - 1/4| \end{cases} \ , \tag{104}$$

and

$$Y_\alpha(x) = \begin{cases} -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{x}\right)^\alpha & ; \quad 0 < x \ll \sqrt{\alpha+1} \\ \\ \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \alpha\pi/2 - \pi/4\right) & ; \quad x \gg |\alpha^2 - 1/4| \end{cases} \ . \tag{105}$$

Both solutions $Z_\alpha = J_\alpha$, Y_α also satisfy the recursion relations

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n \left[x^{-\alpha} Z_\alpha(x)\right] = x^{\alpha-n} Z_{\alpha-n}(x) \tag{106}$$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n \left[x^{-\alpha} Z_\alpha(x)\right] = (-1)^n x^{\alpha-n} Z_{\alpha+n}(x) \ . \tag{107}$$

Many other nice properties of Bessel functions are described in Ref. [50].

Two specific properties of Bessel functions that we will use frequently are orthogonality and completeness. This follows from them being eigenfunctions of a Sturm-Liouville operator. In particular, note that we can rewrite the Bessel equation as

$$\frac{1}{x}\partial_x\left[x\,\partial_x\, f(x)\right] - \frac{\alpha^2}{x^2} f(x) = -f(x) \ . \tag{108}$$

This matches the Sturm-Liouville form with $p(x) = x$, $q(x) = \alpha^2/x^2$, $w(x) = x$, and eigenvalue $\lambda^2 = 1$. Other (positive) eigenvalues arise with the change of variables $x \to \lambda\, x$ since both terms on the left-hand side just pick up a factor of $1/\lambda^2$ while the right-hand side stays the same. Discrete values of λ arise if we also demand that the functions $f(x)$ also satisfy fixed boundary conditions on an interval.

As an example, suppose we impose Dirichlet BCs on the interval $x \in [a, b]$. Solving the equation $Df = -\lambda^2 f$, the most solutions take the form

$$f(x) = c_1 \left[J_\alpha(\lambda x) + c_2 \, Y_\alpha(\lambda x) \right] \ . \tag{109}$$

Imposing BCs at $x = a, b$ then implies

$$c_2 = -J_\alpha(\lambda a)/Y_\alpha(\lambda a) \tag{110}$$

$$0 = J_\alpha(\lambda b) + c_2 \, Y_\alpha(\lambda b) \ . \tag{111}$$

The second equation allows a set of discrete values for $\lambda = \lambda_n$ for which a non-trivial solution exists corresponding to the zeros of the combined sum of Bessel functions. By the Sturm-Liouville theorem, the set of solutions for the set of λ_n forms a complete basis $\{f_n\}$ for the vector space of functions on the interval with Dirichlet BCs. These basis functions satisfy orthonormality (after normalizing them appropriately) in the form

$$\int_a^b dx \; x \, f_m(x) \, f_n(x) = \delta_{mn} \ , \tag{112}$$

which matches the Sturm-Liouville expectation.

References

[1] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge, UK: Univ. Pr. (1998) 402 p

[2] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge, UK: Univ. Pr. (1998) 531 p

[3] D. Tong, "Lectures on String Theory," http://www.damtp.cam.ac.uk/user/tong/string.html.

[4] T. Appelquist, A. Chodos and P. G. O. Freund, "Modern Kaluza-klein Theories," READING, USA: ADDISON-WESLEY (1987) 619 P. (FRONTIERS IN PHYSICS, 65)

[5] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [hep-ph/9803315].

[6] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV," Phys. Lett. B **436**, 257 (1998) [hep-ph/9804398].

[7] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity," Phys. Rev. D **59**, 086004 (1999) [hep-ph/9807344].

[8] L. Randall and R. Sundrum, "A Large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221].

[9] L. Randall and R. Sundrum, "An Alternative to compactification," Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].

[10] N. Arkani-Hamed, S. Dimopoulos and J. March-Russell, "Stabilization of submillimeter dimensions: The New guise of the hierarchy problem," Phys. Rev. D **63**, 064020 (2001) [hep-th/9809124].

[11] J. Polchinski, "Tasi lectures on D-branes," hep-th/9611050.

[12] C. V. Johnson, "D-brane primer," hep-th/0007170.

[13] H. L. Verlinde, "Holography and compactification," Nucl. Phys. B **580**, 264 (2000) doi:10.1016/S0550-3213(00)00224-8 [hep-th/9906182].

[14] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications," Phys. Rev. D **66**, 106006 (2002) doi:10.1103/PhysRevD.66.106006 [hep-th/0105097].

[15] C. Csaki, In *Shifman, M. (ed.) et al.: From fields to strings, vol. 2* 967-1060 [hep-ph/0404096].

[16] C. Csaki, J. Hubisz and P. Meade, "TASI lectures on electroweak symmetry breaking from extra dimensions," hep-ph/0510275.

[17] H. C. Cheng, "2009 TASI Lecture – Introduction to Extra Dimensions," arXiv:1003.1162 [hep-ph].

[18] G. D. Kribs, "TASI 2004 lectures on the phenomenology of extra dimensions," hep-ph/0605325.

[19] D. J. Kapner, T. S. Cook, E. G. Adelberger, J. H. Gundlach, B. R. Heckel, C. D. Hoyle and H. E. Swanson, "Tests of the gravitational inverse-square law below the dark-energy length scale," Phys. Rev. Lett. **98**, 021101 (2007) [hep-ph/0611184].

[20] V. Khachatryan *et al.* [CMS Collaboration], "Search for dark matter, extra dimensions, and unparticles in monojet events in proton-proton collisions at $\sqrt{s} = 8$ TeV," arXiv:1408.3583 [hep-ex].

[21] G. Aad *et al.* [ATLAS Collaboration], "Search for new phenomena in final states with an energetic jet and large missing transverse momentum in pp collisions at $\sqrt{s} = 8$ TeV with the ATLAS detector," arXiv:1502.01518 [hep-ex].

[22] S. Cullen and M. Perelstein, "SN1987A constraints on large compact dimensions," Phys. Rev. Lett. **83**, 268 (1999) [hep-ph/9903422].

[23] C. Cski and P. Tanedo, "Beyond the Standard Model," 2013 European School of High-Energy Physics, Paradfurdo, Hungary, 5 - 18 Jun 2013, pp.169-268 (CERN-2015-004) doi:10.5170/CERN-2015-004.169 [arXiv:1602.04228 [hep-ph]].

[24] H. Davoudiasl, J. L. Hewett and T. G. Rizzo, "Phenomenology of the Randall-Sundrum Gauge Hierarchy Model," Phys. Rev. Lett. **84**, 2080 (2000) doi:10.1103/PhysRevLett.84.2080 [hep-ph/9909255].

[25] G. Aad *et al.* [ATLAS Collaboration], "Search for high-mass dilepton resonances in pp collisions at $\sqrt{s}$ = 8TeV with the ATLAS detector," Phys. Rev. D **90**, no. 5, 052005 (2014) doi:10.1103/PhysRevD.90.052005 [arXiv:1405.4123 [hep-ex]].

[26] K. Agashe, A. Belyaev, T. Krupovnickas, G. Perez and J. Virzi, "LHC Signals from Warped Extra Dimensions," Phys. Rev. D **77**, 015003 (2008) doi:10.1103/PhysRevD.77.015003 [hep-ph/0612015].

[27] B. Lillie, L. Randall and L. T. Wang, "The Bulk RS KK-gluon at the LHC," JHEP **0709**, 074 (2007) doi:10.1088/1126-6708/2007/09/074 [hep-ph/0701166].

[28] G. Aad *et al.* [ATLAS Collaboration], "A search for $t\bar{t}$ resonances using lepton-plus-jets events in proton-proton collisions at $\sqrt{s}$ = 8 TeV with the ATLAS detector," JHEP **1508**, 148 (2015) doi:10.1007/JHEP08(2015)148 [arXiv:1505.07018 [hep-ex]].

[29] J. Thaler and L. T. Wang, "Strategies to Identify Boosted Tops," JHEP **0807**, 092 (2008) doi:10.1088/1126-6708/2008/07/092 [arXiv:0806.0023 [hep-ph]].

[30] D. E. Kaplan, K. Rehermann, M. D. Schwartz and B. Tweedie, "Top Tagging: A Method for Identifying Boosted Hadronically Decaying Top Quarks," Phys. Rev. Lett. **101**, 142001 (2008) doi:10.1103/PhysRevLett.101.142001 [arXiv:0806.0848 [hep-ph]].

[31] M. Carena, A. Delgado, E. Ponton, T. M. P. Tait and C. E. M. Wagner, "Precision electroweak data and unification of couplings in warped extra dimensions," Phys. Rev. D **68**, 035010 (2003) doi:10.1103/PhysRevD.68.035010 [hep-ph/0305188].

[32] K. Agashe, A. Delgado, M. J. May and R. Sundrum, "RS1, custodial isospin and precision tests," JHEP **0308**, 050 (2003) doi:10.1088/1126-6708/2003/08/050 [hep-ph/0308036].

[33] K. Agashe, R. Contino, L. Da Rold and A. Pomarol, "A Custodial symmetry for $Zb\bar{b}$," Phys. Lett. B **641**, 62 (2006) doi:10.1016/j.physletb.2006.08.005 [hep-ph/0605341].

[34] C. Csaki, A. Falkowski and A. Weiler, "The Flavor of the Composite Pseudo-Goldstone Higgs," JHEP **0809**, 008 (2008) doi:10.1088/1126-6708/2008/09/008 [arXiv:0804.1954 [hep-ph]].

[35] M. Bauer, S. Casagrande, U. Haisch and M. Neubert, "Flavor Physics in the Randall-Sundrum Model: II. Tree-Level Weak-Interaction Processes," JHEP **1009**, 017 (2010) doi:10.1007/JHEP09(2010)017 [arXiv:0912.1625 [hep-ph]].

[36] W. D. Goldberger and M. B. Wise, "Modulus stabilization with bulk fields," Phys. Rev. Lett. **83**, 4922 (1999) doi:10.1103/PhysRevLett.83.4922 [hep-ph/9907447].

[37] W. D. Goldberger and M. B. Wise, "Phenomenology of a stabilized modulus," Phys. Lett. B **475**, 275 (2000) doi:10.1016/S0370-2693(00)00099-X [hep-ph/9911457].

[38] C. Csaki, M. Graesser, L. Randall and J. Terning, "Cosmology of brane models with radion stabilization," Phys. Rev. D **62**, 045015 (2000) doi:10.1103/PhysRevD.62.045015 [hep-ph/9911406].

[39] C. Csaki, M. L. Graesser and G. D. Kribs, "Radion dynamics and electroweak physics," Phys. Rev. D **63**, 065002 (2001) doi:10.1103/PhysRevD.63.065002 [hep-th/0008151].

[40] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] doi:10.1023/A:1026654312961 [hep-th/9711200].

[41] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[42] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[43] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) doi:10.1016/S0370-1573(99)00083-6 [hep-th/9905111].

[44] N. Arkani-Hamed, M. Porrati and L. Randall, "Holography and phenomenology," JHEP **0108**, 017 (2001) doi:10.1088/1126-6708/2001/08/017 [hep-th/0012148].

[45] R. Rattazzi and A. Zaffaroni, "Comments on the holographic picture of the Randall-Sundrum model," JHEP **0104**, 021 (2001) doi:10.1088/1126-6708/2001/04/021 [hep-th/0012248].

[46] T. Gherghetta, "A Holographic View of Beyond the Standard Model Physics," doi:10.1142/9789814327183_0004 arXiv:1008.2570 [hep-ph].

[47] P. H. Ginsparg, "Applied Conformal Field Theory," hep-th/9108028.

[48] Y. Nakayama, "Scale invariance vs conformal invariance," Phys. Rept. **569**, 1 (2015) doi:10.1016/j.physrep.2014.12.003 [arXiv:1302.0884 [hep-th]].

[49] T. Banks and A. Zaks, "On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions," Nucl. Phys. B **196**, 189 (1982). doi:10.1016/0550-3213(82)90035-9

[50] Bessel functions are discussed in many textbooks, but this link is easy to remember: https://en.wikipedia.org/wiki/Bessel_function