# PHYS 528 Homework #9

Due: Mar.23, 2017

1. Linear and non-linear sigma models.

   Consider a theory of four real scalars written in terms of a complex ($2\times 2$) matrix-valued field $S$ of the form

   $$S = \sigma(x)\mathbb{I}/2 + i\alpha^a(x)t^a,$$

   where $\mathbb{I}$ is the ($2 \times 2$) unit matrix, $t^a = \sigma^a/2$, and $\sigma$, $\alpha^a$ are real scalars. Take the Lagrangian to be

   $$\mathscr{L} = tr(\partial_\mu S^\dagger \partial^\mu S) - \frac{\lambda}{2}\left[tr(S^\dagger S) - v^2/2\right]^2 .$$

   This renormalizable theory is an example of a *linear sigma model.*

   a) Show that this theory has a global $SU(2)_L \times SU(2)_R$ symmetry under which the matrix field transforms as $S \to S' = LSR^\dagger$.

   b) Show that one of the global minima of the potential is

   $$\langle S \rangle = (v/2)\,\mathbb{I}\ .$$

   What fraction of the original symmetry does this spontaneously break? How many NGBs will there be?

   c) Let us choose to expand around this "identity" vacuum state. One way to do so is write

   $$S(x) = [v + h(x)]/2 + i\alpha^a(x)t^a\ .$$

   Check that the kinetic terms are canonical and find the masses of these fields.

   d) Another way to expand around the vacuum is to write

   $$S(x) = \frac{1}{2}[v + H(x)]\exp[2i\pi^a(x)t^a/f]\ ,$$

   for some $f$ with mass dimension one. The exponential term can be thought of as a general transformation on the identity vacuum. For this choice of field variables:

   i) Work out the kinetic terms for these new field variables, and fix $f$ such that they become canonical. What are the masses of the $H$ and $\pi^a$ excitations?

   ii) Suppose we are interested in physics at energies well below the mass of the $H$ field. Work out the leading terms in the low-energy EFT you get by integrating out $H$. The result is sometimes called a *non-linear sigma model.*

e) Couple $S$ to a doublet of Dirac fermions,

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix},$$

with $\psi_L \to L\psi_L$ and $\psi_R \to R\psi_R$ under $SU(2)_L \times SU(2)_R$. These symmetries forbid bare fermion masses, but we can still generate masses consistent with the symmetries by coupling to $S$:

$$-\mathscr{L} \supset g\,\bar{\psi}_L S \psi_R + h.c.$$

Work out the couplings in terms of the exponential representation of $S$ discussed above. What are the resulting fermion masses? As the notation here might suggest, this theory was invented as an early model for the interactions of pions and nucleons.

2. Chiral perturbation theory (with two massless quarks).

a) Work out the transformation properties of the $\Pi$ fields to linear order in $c_V^a$ and $c_A^a$ for the two cases $c_A^a = 0$ ($SU(2)_V$) and $c_V^a = 0$ (broken generators).

b) Expand the leading ($\mathcal{O}(p^2)$) term in the chiral perturbation theory Lagrangian to quartic order in the $\Pi$ fields. Show that you get a canonical kinetic term and an interaction among pions.
*Hint:* $\sigma^a\sigma^b = \delta^{ab}\mathbb{I} + i\epsilon^{abc}\sigma^c$.

c) Find the Noether currents for $SU(2)_L$, $SU(2)_R$, and $U(1)_V$ in the low-energy EFT. Expand these to leading non-trivial order in the $\Pi$ fields.
*Hint: treat* $\Sigma$ *and* $\Sigma^\dagger$ *as the dynamical fields.*

d) Expand out the term

$$-\mathscr{L} \supset \frac{1}{2}\tilde{\Lambda}^3 tr(M\Sigma) + h.c\ ,$$

where $M = diag(m_u, m_d)$, and derive mass terms for the pions.
*Hint: recall that* $\pi^- = (\pi^+)^*$ *is complex, and that the mass term for a complex scalar does not have a* $1/2$ *factor.*

3. Compute the decay widths of $\pi^- \to \mu\bar{\nu}_\mu$ and $\pi^- \to e\bar{\nu}_e$ using the effective Hamiltonian discussed in the the notes. Compare your result to the measured values and use this to extract a numerical value for $f_\pi$.