# PHYS 528 Homework #8

Due: Mar.16, 2017

1. Symmetries, renormalization, and EFTs.

   Consider (again) the basic Yukawa theory of a real scalar $\phi$ and a Dirac fermion $\psi$:

   $$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{A}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \overline{\psi}i\gamma^\mu\partial_\mu\psi - M\overline{\psi}\psi - y\phi\overline{\psi}\psi \ .$$

   For $A = 0$ and $y = 0$ this theory has a symmetry under $\phi \to -\phi$, while for $M = 0$ and $y = 0$ the theory has symmetry under $\psi \to i\gamma^5\psi$, and for $M = 0$ and $A = 0$ it is symmetric under $\phi \to -\phi$ and $\psi \to i\gamma^5\psi$ together.

   a) Show that these symmetries can all be restored individually for non-zero $M$, $A$, and $y$ if we allow these parameters to transform as well.

   b) Check that the leading expressions for $\Delta\Gamma^{(\bar{\psi}\psi)}$ and $\Delta\Gamma^{(\phi^3)}$ in the notes due to $y \neq 0$ are consistent with the restoration of these symmetries for non-zero $M$ and $y$ if we also allow these parameters to transform appropriately.

   c) Show that even if we started with $\lambda = 0$, a non-zero value is generated at one-loop order by the $y$ coupling. Check that the the expression for this ($\Delta\Gamma^{(\phi^4)}$) is consistent with the restoration of the symmetries above when $y$ transforms too.

   d) For $\sqrt{m^2} \gg M$, we can integrate out the massive scalar. A trick to obtain the tree-level EFT Lagrangian is to replace $\phi$ with the solution of its equation of motion in the full Lagrangian. Do this for $A \to 0$ and $\lambda \to 0$.
   *Hint: we need $p^2 \ll m^2$ to use the EFT, and this implies $\partial^2\phi \ll m^2\phi$.*

2. Compute all the gauge anomaly coefficients for the SM. Show that they vanish independently for each generation.
   *Hint: for $SU(2)_L^3$, use $t_L^a = \sigma^a/2$ and the anticommutator for sigma matrices.*

3. Operator scaling and the renormalization group.

   We are often interested in using theories defined at one scale to make predictions at a very different energy scales. In this case it is often very helpful to extrapolate from one scale to another using the renormalization group (RG). Let us assume that we start at scale $\mu_0$ and extrapolate to scale $\mu$ in a theory whose interactions are dominated by gauge interactions of strength $\alpha(\mu) = g^2(\mu)/4\pi$.

   a) Suppose the evolution of the gauge coupling is given by

   $$\frac{d\alpha}{dt} = -b\,\alpha^2(t) \ ,$$

   where $t = \ln(\mu/\mu_0)$ and $b$ is a non-zero constant. Solve this equation for $\alpha(\mu)$ in terms of $\alpha(\mu_0)$.
   *Hint: find the differential equation for $\alpha^{-1}(\mu)$ and solve it instead.*

b) The gauge coupling is the coefficient of just one operator in the effective Lagrangian; we can also apply RG to other operators. Suppose we have

$$-\mathscr{L} \supset \zeta\, M^{4-n} \mathcal{O}_\zeta^{(n)} \ ,$$

where $\mathcal{O}_\zeta^{(n)}$ is an operator with a (classical) mass dimension of $n$, $\zeta$ is dimensionless, and $M$ is a fixed dimension-one constant. Just like the gauge coupling, the coefficient $\zeta$ can be RG-evolved. Suppose its evolution equation is

$$\frac{1}{\zeta}\frac{d\zeta}{dt} = \gamma_\zeta(\alpha(t)) \ ,$$

for some function $\gamma_\zeta$. Show that a solution to this equation for $\zeta(\mu)$ is

$$\zeta(\mu) = \zeta(\mu_0)\ \exp\left[\int_{t_0}^{t} dt'\, \gamma_\zeta(\alpha(t'))\right] \ .$$

c) For processes with typical momentum $p$, the relative contribution of the operator $\mathcal{O}_\zeta^{(n)}$ to the dynamics is

$$\zeta(\mu = p)\ \left(\frac{p}{M}\right)^{n-4}$$

For this reason, we say that $n$ is the classical scaling dimension of the operator $\mathcal{O}_\zeta^{(n)}$. However, since quantum effects lead to the RG evolution of $\zeta$, the full quantum evolution is a bit different.

i) Assume that

$$\gamma_\zeta(t) = -a\,\alpha(t) \ ,$$

for some constant $a$. Use this and your solution for $\alpha(\mu)$ to solve for $\zeta(\mu)$ in terms of $\zeta(\mu_0)$ and $\alpha(\mu_0)$. What does this imply for the net momentum scaling of the operator?
*Hint: you know both $d\alpha/dt$ and $\gamma_\zeta$ as functions of $\alpha$.*

ii) Suppose instead that the theory is approximately *conformal*, in the sense that $\alpha(\mu) = \alpha_*$ is constant. Show that this implies that $\gamma_\zeta = \gamma_*$ is constant as well, and solve for $\zeta(\mu)$ in this case. What does this imply for the momentum scaling of the operator?

d) Suppose we include higher-order corrections to the RG equation for $\alpha$, and that these take the form

$$\frac{d\alpha}{dt} = -b_1\alpha^2 + b_2\alpha^3 \ ,$$

with $b_1$, $b_2 > 0$. For what special non-zero value of $\alpha = \alpha_*$ does $d\alpha/dt$ vanish? What happens if we start at $\mu_0 > \mu$ with $\alpha(\mu_0) > \alpha_*$ and evolve down to lower energies ($\mu$ values)? Similarly, what happens if we start at $\mu_0 > \mu$ with $\alpha(\mu_0) < \alpha_*$ and evolve down? The value $\alpha_*$ is sometimes called a non-trivial IR fixed point.