PHYS 528 Homework #3

Due: Feb. 2, 2017

1. Connections.

   Consider a fermion in an Abelian gauge theory transforming according to $\psi(x) \to e^{i\alpha(x)}\psi(x) = U(x)\psi(x)$. Suppose there exists an object $P(x, y)$ with $P(x, x) = 1$ and the transformation property $P(x, y) \to U(x)P(x, y)U^{-1}(y)$ for any pair of points x and y. Clearly, $P(x, y)\psi(y) \to U(x)[P(x, y)\psi(y)]$ transforms in the same way as $\psi(x)$ (rather than $\psi(y)$).

   a) Define a derivative-like operator $\tilde{D}_\mu$ according to

   $$n^\mu \tilde{D}_\mu \psi(x) = \lim_{\epsilon \to 0} \left[P(x, x + \epsilon n)\psi(x + \epsilon n) - \psi(x)\right]/\epsilon$$

   where n^μ is any unit 4-vector. The idea behind this definition is that it only really makes sense to take the difference of two objects with the same transformation properties. As far as working out this operator goes, we can expand $P(x, x+\epsilon n) = [1 + i\epsilon n^\mu \tilde{A}_\mu(x)]$ and drop all the higher-order terms in ϵ. Work out $\tilde{D}_\mu$ in terms of ordinary derivatives and the function $\tilde{A}_\mu(x)$. Does this remind you of anything?

   b) Under an arbitrary gauge transformation,

   $$P(x, x + \epsilon n) \to U(x)P(x, x + \epsilon n)U^{-1}(x + \epsilon n) \equiv [1 + i\epsilon n^\mu \tilde{A}'_\mu(x)].$$

   Evaluate $\tilde{A}'_\mu$ in terms of $\tilde{A}_\mu$ and α. Again, does this remind you of anything?

2. Charged scalars.

   In class we showed how to make a theory with a charged fermion invariant under local $U(1)$ gauge transformations. One can do the same thing for a theory with a complex scalar whose Lagrangian is invariant under global $U(1)$ phase transformations.

   a) Starting with a Lagrangian for the complex scalar with a canonical kinetic term and a potential that depends only on $|\phi|^2$, work out the symmetry current corresponding to the symmetry under global phase rotations.

   b) Find a way to make this theory invariant under local $U(1)$ gauge transformations by adding a vector gauge field that has the same transformation properties as in the fermion case we discussed in class.

   c) Isolate the coupling of the gauge field to the scalar and compare the scalar portion of the operator to the result of part a).

3. Non-Abelian gauge invariance.

   a) Work out the details and show explicitly that the covariant derivative we discussed for the non-Abelian case transforms according to $D_\mu\psi \to U_r(D_\mu\psi)$.
   Hint: $0 = \partial_\mu(\mathbb{I}) = \partial_\mu(U_r U_r^{-1}) = \partial_\mu(U_r^{-1} U_r)$ *with* $U_r^{-1} = U_r^\dagger$.

b) We had that $A_{r\,\mu} := A^a_\mu t^a_r \to A'^{\,a}_{r\,\mu} t^a_r = U_r A_{r\,\mu} U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1}$. For $U_r = e^{i\alpha^a t^a_r}$, work out the corresponding transformation of the coefficient functions A^a_μ to linear order in the α^a parameters to derive the result of Eq. (9) in `notes-03` explicitly. Does this result depend on the specific representation chosen? (*i.e.* would the same transformation of the A^a_μ coefficients also work for other representations?)

c) Fill in the details of the derivation of $[D_\mu, D_\nu]\psi = igt^a_r(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu)\psi$, for ψ transforming under the rep r of the gauge group.

d) Write out the covariant derivative acting on a field transforming under the adjoint rep of the non-Abelian group G in terms of the structure constants f^{abc}.

4. Scalar decay to vectors.

Consider the interaction

$$-\mathscr{L} \supset \frac{1}{\Lambda}\, h\, V_{\mu\nu} V^{\mu\nu} \ ,$$

where h is a real scalar of mass m_h, $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ for some vector boson V_μ, and $\Lambda \gg m_h$ is some very large mass scale. This interaction allows the decay $h(p) \to V_\mu(k_1) + V_\nu(k_2)$, for which the amplitude is

$$-i\mathcal{M} = -\frac{2i}{\Lambda}(k_1^\mu \epsilon_1^\nu - k_1^\nu \epsilon_1^\mu)(k_2^\alpha \epsilon_2^\beta - k_2^\beta \epsilon_2^\alpha)\eta_{\alpha\mu}\eta_{\beta\nu} \ ,$$

where ϵ_1 and ϵ_2 refer to the polarizations of two outgoing vectors.

a) If V_μ is massless, there are two physical polarization states and a built-in gauge invariance. Compute the summed and squared matrix element "$|\mathcal{M}|^2$" relevant for the total unpolarized decay rate in the h rest frame using the partial completeness relation $\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{\nu\,*}(p,\lambda) = -\eta^{\mu\nu} + (stuff\ you\ can\ ignore)$, just like what we used for external photons in QED.

b) A second way to compute the summed and squared matrix element "$|\mathcal{M}|^2$" is to specify external polarization vectors and add up the results. Do this here using the explicit polarization vectors discussed in `notes-03` and summing over all the possibilities.
Hint: since the initial state is at rest and has no spin, you can choose the $\hat{z}$ axis to lie along the direction of the first outgoing vector, $\vec{k}_1 = \|\vec{k}_1\|\,\hat{z}$.

c) Suppose instead that the vector V_μ is massive, with mass m_V. This implies that it has three physical polarization states. The corresponding polarization 4-vectors should satisfy

$$\epsilon(p,\lambda)\cdot\epsilon^*(p,\lambda') = -\delta_{\lambda\lambda'} \ , \qquad p\cdot\epsilon = 0 \ .$$

For $\vec{p} = \|\vec{p}\|\,\hat{z}$, find a set of three 4-vectors that satisfy these conditions. You should be able to identify two of them as *transverse*, and one as *longitudinal.*

d) Use these three polarization 4-vectors to compute the summed and squared matrix element "$|\mathcal{M}|^2$" for $h \to V_\mu V_\nu$ in the rest frame of the decaying scalar. Also, compare the squared matrix element for longitudinal final states to those for transverse final states.