PHYS 528 Homework #2

Due: Jan.26, 2017

1. Do #5 originally listed on HW#1:

   Consider a massive Z' vector boson that couples to electrons with a vertex factor equal to $-ig'\gamma^\mu$.

   a) A massive vector has three independent polarization states. These can represented by any three independent unit 4-vectors $\epsilon_\mu(p, \lambda)$ satisfying the constraints $p^\mu \epsilon_\mu = 0$ and $\epsilon^*_\mu(p, \lambda)\epsilon^\mu(p, \lambda') = -\delta_{\lambda\lambda'}$, where p^μ is the four-momentum of the vector boson and $\lambda = 1, 2, 3$ labels the three different polarizations. Find a simple set of polarization vectors in the rest frame of the massive vector. Show that they satisfy the constrained completeness relation

   $$\sum_\lambda \epsilon_\mu(p, \lambda)\epsilon^*_\nu(p, \lambda) = -\eta_{\mu\nu} + p_\mu p_\nu / m^2_{Z'}.$$

   b) Compute the total unpolarized decay width for $Z' \to e^+e^-$. For this, use the vertex and the completeness relation stated above. Keep the full dependence on the masses of the Z' and the electron.

2. Complex scalar theory.

   $$\mathscr{L} = |\partial\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \; . \tag{1}$$

   For this theory:

   a) Find the vertex factor for the interaction.
   Hint: treat ϕ and ϕ^* as independent variables.

   b) Derive the equations of motion and the conserved current for the symmetry $\phi \to e^{i\alpha}\phi$. Show explicitly that the current really is conserved.

   c) We identify ϕ with a scalar and ϕ^* with its antiparticle, having the same mass but the opposite charge. Feynman rules for a complex scalar are similar to those of a real scalar, but it is necessary to keep track of the difference between ϕ and ϕ^*. To do so, we add arrows to complex scalar lines to track the direction of charge flow. In particular, a ϕ field leads to an arrow pointing into a vertex, and a ϕ^* field gives an arrow pointing out of a vertex. The external leg for a particle in the initial state gets an arrow pointing in to the diagram, and a particle in the final state gets an arrow pointing out of the diagram. For antiparticles, the arrow directions are reversed. (This works just like for fermions.) To make an allowed Feynman diagram, the external legs and the legs on vertices must be connected up such that the arrows point in the same direction. Use all this to draw the leading-order Feynman diagrams for $\phi\phi \to \phi\phi$, $\phi^*\phi^* \to \phi^*\phi^*$, and $\phi\phi \to \phi\phi^*$, making sure to draw the arrows correctly, and compute their cross sections.

3. Consider the theory discussed in ***e.g.* 3.** of `notes-02`:

$$\mathscr{L} = |\partial\phi|^2 - M^2|\phi|^2 + \sum_{i=1}^{2} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - (y\phi\bar{\psi}_1\psi_2 + h.c.) \ . \tag{2}$$

For this theory:

a) Find the equations of motion for ϕ, ψ_1, and ψ_2.

b) Work out the conserved current for the rephasing symmetry discussed in the note.

c) Show that this current really is conserved.
Hint: make use of the equations of motion.

d) Derive the interaction vertices in the theory, and use them to draw the leading-order Feynman diagrams for the processes $\phi + \phi^* \to \psi_1 + \bar{\psi}_1$ and $\phi \to \psi_2\bar{\psi}_1$. In both cases, compute the net symmetry charge of the initial and final states.

4. Adjoint representation of $SU(2)$.

a) Work out the 3×3 adjoint representation matrices using the structure constants and show that they are Hermitian and obey the right commutation relations.

b) For these matrices, compute $T_2(A)$ and $C_2(A)$.

c) Find the spin matrices S_x, S_y, and S_z for a $s = 1$ system in quantum mechanics using the usual raising and lowering operator tricks. Compute T_2 and C_2 for them. These also give a representation of the adjoint of $SU(2)$, but in a different basis compared to the one in parts a) and b).
Hint: recall that $S_\pm = S_x \pm iS_y$ *and* $S_\pm\,|s, m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\;|s,\, m\pm 1\rangle$

5. Lie algebra stuff.

a) Show that the structure constants are completely antisymmetric for any Lie group.
Hint: the first two indices are antisymmetric by their definition. For the last one, multiply the fundamental commutation relation by t^d*, take a trace, and keep in mind that traces are cyclic.*

b) Show that for any rep $\{t_r^a\}$, the set of matrices $\{-(t_r^a)^*\}$ also gives a rep.

c) Prove that the adjoint rep of a compact Lie group is always a real representation.

d) Prove that the Casimir T_r^2 commutes with any generator t_r^b for any rep r.
Hint: commute through with structure constants, and then use their antisymmetry.

e) Show that for an irrep of a simple group, $d(r)C_2(r) = d(A)T_2(r)$.
Hint: contract the Dynkin index equation with δ^{ab}*.*