PHYS 528 Lecture Notes #6

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1 Flavour in the Standard Model

In our previous discussion of the SM we did not say anything about the three families of quarks and leptons. There is a very interesting story here, and we turn to it now. The SM contains three copies of all the fermion representations that we call *families*, *generations*, or *flavours*. They are

$$\text{Quarks}: \quad \begin{cases} u_{L,R} & c_{L.R} & t_{L,R} & Q=+2/3 \\ d_{L,R} & s_{L,R} & b_{L,R} & Q=-1/3 \end{cases} \tag{1}$$

$$\text{Leptons}: \quad \begin{cases} \nu_{e_L} & \nu_{\mu_L} & \nu_{\tau_L} & Q=\ \ 0 \\ e_{L,R} & \mu_{L,R} & \tau_{L,R} & Q+-1 \end{cases} \tag{2}$$

The first column corresponds to the first generation (or family), the second column to the second generation, and the third column to the third. The elements of each generation have identical sets of $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum numbers (*i.e.* representations) but, as we will see shortly, differ greatly in their masses.

Instead of writing out all three generations explicitly, it is much easier to use a condensed notation with a generation index $A = 1, 2, 3$. For example, we will write u_{R_A} where

$$u_{R_{A=1}} = u_R, \quad u_{R_2} = c_R, \quad u_{R_3} = t_R, \tag{3}$$

and similarly for the other states. Since all three generations have identical quantum numbers, we can choose our field variables such that all the gauge-covariantized kinetic terms are diagonal in generation space.[1] That is

$$\mathscr{L}_{gauge} \supset \bar{Q}_{L_A} i\gamma^\mu D_\mu Q_{L_A} + \bar{u}_{R_A} i\gamma^\mu D_\mu u_{R_A} + \ldots \tag{4}$$

This choice of field variables is sometimes called the *gauge eigenbasis*. We will always implicitly start off with this basis and work from there.

Going back to the Yukawa interactions, we see that gauge invariance allows them to have a non-trivial family-mixing structure. Put another way, the most general set of gauge-invariant Yukawa terms we can write (taking the generational structure into account) is

$$\begin{aligned} -\mathcal{L}_{Yukawa} &= y_{u_{AB}} \bar{Q}_{L_A} \tilde{\Phi}\, u_{R_B} + y_{d_{AB}} \bar{Q}_{L_A} \Phi\, d_{R_B} + y_{e_{AB}} \bar{L}_{L_A} \Phi\, e_{R_B} + (h.c.) \\ &= (v + h/\sqrt{2}) \bar{u}_{L_A} y_{u_{AB}} u_{R_B} + (v + h/\sqrt{2}) \bar{d}_{L_A} y_{d_{AB}} d_{R_B} + (v + h/\sqrt{2}) \bar{e}_{L_A} y_{e_{AB}} e_{R_B} + (h.c.) \\ &= (v + h/\sqrt{2})\, \bar{u}_L y_u u_R + (v + h/\sqrt{2})\, \bar{d}_L y_d d_R + (v + h/\sqrt{2})\, \bar{e}_L y_e e_R + (h.c.) \end{aligned} \tag{5}$$

[1] If they aren't to begin with, just make rotations and rescalings in generation space such that they are. This is consistent with gauge invariance since these rotations only mix fields that transform under identical gauge representations.

In the third line we have implicitly contracted the generation indices and we have written this expression in terms of matrices and row and column vectors in generation space.

The portions of the expression above involving v are mass matrices for the up- and down-type quarks and the charged leptons. In general they are non-diagonal if the Yukawa couplings tie different generations together; more precisely, they are 3×3 complex matrices. To do perturbation theory, we should diagonalize them. Any complex matrix can always be diagonalized by a pair of unitary matrices. To achieve this, define a new set of fields according to

$$\begin{aligned}
u_{L_A} &= V^L_{u_{AB}}\, u'_{L_B}, \qquad u_{R_A} = V^R_{u_{AB}}\, u'_{R_B} \\
d_{L_A} &= V^L_{d_{AB}}\, d'_{L_B}, \qquad d_{R_A} = V^R_{d_{AB}}\, d'_{R_B} \\
\\
e_{L_A} &= V^L_{e_{AB}}\, e'_{L_B}, \qquad e_{R_A} = V^R_{e_{AB}}\, e'_{R_B} \\
\nu_{L_A} &= V^L_{\nu_{AB}}\, \nu'_{L_B},
\end{aligned} \tag{6}$$

Here, $V_f^{L,R}$ are all unitary matrices. We can choose them such that they will bi-diagonalize the Yukawa interaction matrices. That is

$$\begin{aligned}
V_u^{L^\dagger} y_u V_u^R &= \frac{1}{v}\, diag(m_u, m_c, m_t) \\
V_d^{L^\dagger} y_d V_d^R &= \frac{1}{v}\, diag(m_d, m_s, m_b) \\
V_e^{L^\dagger} y_e V_e^R &= \frac{1}{v}\, diag(m_e, m_\mu, m_\tau)
\end{aligned} \tag{7}$$

In terms of the primed fields, the Yukawa interactions containing the mass terms are now diagonal. For example

$$\begin{aligned}
-\mathscr{L}_{Yukawa} &\supset (v + h/\sqrt{2})\, u_L y_u u_R \\
&= (v + h/\sqrt{2})\, \bar{u}'_L\, (V_u^{L^\dagger} y_u V_u^R)\, u'_R \\
&= (1 + h/\sqrt{2}v)\, (m_u\, \bar{u}'_L u'_R + m_c\, \bar{c}'_L c'_R + m_t\, \bar{t}'_L t'_R)\,.
\end{aligned} \tag{8}$$

Since these field transformations are unitary, the fermion kinetic terms retain their generation-diagonal form. For instance,

$$\begin{aligned}
\bar{Q}_L i\gamma^\mu \partial_\mu Q_L &\to \bar{u}'_L V_u^{L^\dagger} i\gamma^\mu \partial_\mu V_u^L u'_L + \bar{d}'_L V_d^{L^\dagger} i\gamma^\mu \partial_\mu V_d^L d'_L \\
&= \bar{Q}'_L i\gamma^\mu \partial_\mu Q'_L
\end{aligned} \tag{9}$$

These keep the same form because the kinetic terms only have LL and RR pieces and do not mix the upper and lower components of the $SU(2)_L$ doublets. As a result, we always get the combination $V_f^{L,R^\dagger} V_f^{L,R} = \mathbb{I}$. The primed field basis we have defined therefore has canonical kinetic terms and diagonal masses, and is therefore a good basis to use for doing perturbation theory. This basis is often called the *mass eigenbasis*.

Let us turn next to the couplings of the primed fields to the bosons of the theory. By construction, or from Eq. (8), we see that the couplings of the primed fields to the Higgs boson

h are all generation-diagonal. The couplings of fermions to the photon A_μ, the massive Z_μ vector, and the gluon G^a_μ are also diagonal in generation space. This comes about for exactly the same reason that the fermion kinetic terms remain diagonal - the unitary transformations cancel each other out. Things are more interesting for the couplings of fermions to the massive $W^\pm_\mu$ vectors. Here we have

$$\begin{aligned} -\mathscr{L} &\supset \frac{g}{\sqrt{2}}\,\bar{u}_L\gamma^\mu W^+_\mu d_R + \frac{g}{\sqrt{2}}\,\bar{\nu}_L\gamma^\mu W^+_\mu e_R + (h.c.) \\ &= \frac{g}{\sqrt{2}}\,\bar{u}'_L(V^{L\dagger}_u V^L_d)\gamma^\mu W^+_\mu d'_R + \frac{g}{\sqrt{2}}\,\bar{\nu}'_L(V^{L\dagger}_\nu V^L_e)\gamma^\mu W^+_\mu e'_R + (h.c.) \end{aligned} \tag{10}$$

The unitary generation-space matrix appearing in the quark term is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix,

$$V^{(CKM)} = V^{L\dagger}_u V^L_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{11}$$

This represents a physical cross-generational mixing. For the leptons, on the other hand, we can always choose the neutrino mixing matrix $V^L_\nu = V^L_e$ without changing anything else in the SM Lagrangian.[2] Thus we can choose our field variables such that the couplings of the W to leptons remains generation-diagonal. The only physical source of flavour mixing in the SM is therefore the CKM matrix.

Mixing of generations is observed experimentally and seems to be consistent with the CKM picture. Numerically, the magnitudes of the entries in the CKM matrix are

$$|V^{(CKM)}| \simeq \begin{pmatrix} 0.9738 & 0.226 & 0.0043 \\ 0.23 & 0.96 & 0.042 \\ 0.0074 & . & . \end{pmatrix}. \tag{12}$$

The number of decimal places here corresponds approximately to the current experimental precision.

The Yukawa couplings we began with (in the gauge eigenbasis) in Eq. (5) can be complex. This leads to complex phases in the CKM matrix. In general, one can write a 3×3 unitary matrix in terms of three rotation angles ($O(3) \subset SU(3)$) and six phases. Five of these phases can be removed by field redefinitions that leave the real, diagonal form of the mass and kinetic terms unchanged. The remaining phase is physical, and gives rise to observable CP violation. We will discuss this later on in the course.

2 Computing with the Standard Model

When computing within the SM it is customary to work in the mass eigenstate basis, and we will follow this custom. To simplify the notation, we will drop the primes on these states

[2] This would not be true if it were possible to write a mass term for the neutrinos in the SM.

that we had been using to distinguish them from the gauge eigenstates. It is also customary to assemble the 2-component SM fermions into 4-component objects. Thus, we write

$$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \quad d = \begin{pmatrix} d_L \\ d_R \end{pmatrix}, \quad e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}, \quad \nu_L = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix}. \tag{13}$$

Since the W and Z vectors coupled differently to the L and R components, we will have to insert chiral projectors P_L and P_R into the Feynman rules.

The propagators for the SM fermions and the Higgs boson are the same as we had before - see Fig. 1. For vectors, the propagators are (for momentum p)

$$A_\mu \to A_\nu : \qquad \frac{i}{p^2}(-\eta_{\mu\nu}) \tag{14}$$

$$G^a_\mu \to G^b_\nu : \qquad \frac{i}{p^2}\left[-\eta_{\mu\nu} + (1-\xi)p_\mu p_\nu/p^2\right] \tag{15}$$

$$Z_\mu \to Z_\nu : \qquad \frac{i}{p^2 - m_Z^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2\right) \tag{16}$$

$$W^\pm_\mu \to W^\pm_\nu : \qquad \frac{i}{p^2 - m_W^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_W^2\right) \tag{17}$$

The factor ξ in the gluon propagator depends on the choice of gauge and should cancel out of any physically observable quantity. The $W^\pm$ and Z^0 propagators correspond specifically to our choice of *unitary gauge*, and they describe the propagation of a massive vector.[3]

Spin polarization factors for external fermion lines in a Feynman diagram are identical to those we discussed for QED and general non-Abelian gauge theories. External vector lines pick up a polarization vector $\epsilon(p, \lambda)$ as shown in Fig. 1, where p is the momentum of the vector and λ labels the polarization state. Massive and massless vectors have different numbers of polarization states. The massless photon and gluons have *two* physical transverse polarizations. With a massive vector, we can transform to the rest frame in which case the polarizations coincide with the independent states of a spin-1 system: thus a massive vector has *three* distinct polarizations. For example, $\{\epsilon_\mu(p,\lambda)\} = \{(0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)\}$ in the rest frame. Independent of whether a vector is massive or massless, we always have

$$\epsilon_\mu(p,\lambda)\, p^\mu = 0. \tag{18}$$

These properties have important consequences for evaluating Feynman diagrams. In many cases we only care about the unpolarized cross-section, where the final polarizations are summed over and the initial polarizations are averaged. In both cases one encounters spin

[3] In other gauges, they can take a different form.

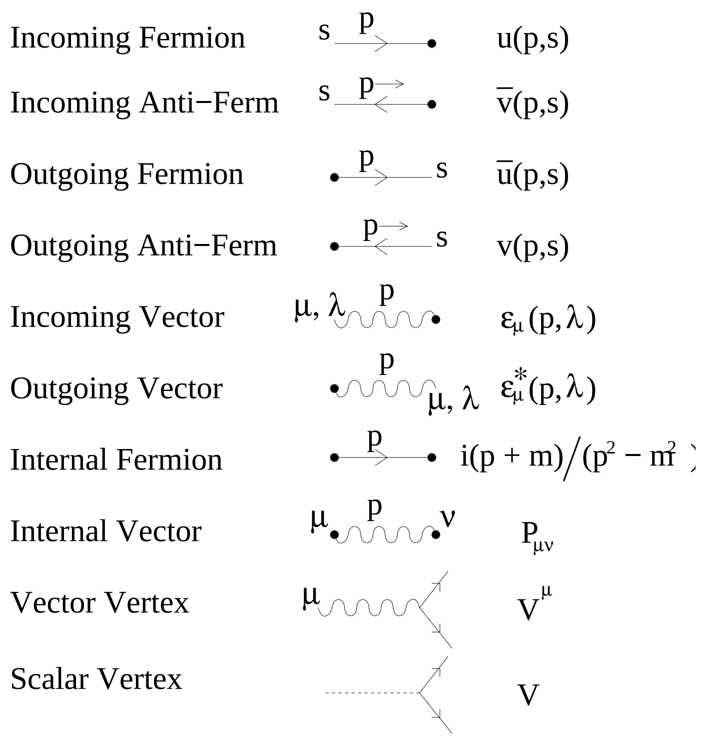


Figure 1: Feynman rules for the Standard Model.

sums that can be simplified. For the SM vectors, we have

$$\text{For } W_\mu,\ Z_\mu: \qquad \sum_{\lambda=1}^{3} \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda) = -\eta_{\mu\nu} + p_\mu p_\nu/m^2 \tag{19}$$

$$\text{For } A_\mu: \qquad \sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can ignore}) \tag{20}$$

$$\text{For } G_\mu: \qquad \sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can't ignore}) \tag{21}$$

The non-ignorable stuff for the gluon polarization sum is related to the presence of non-decoupling ghost fields in the theory. In this case, it usually easiest to choose an explicit set of transverse polarization vectors satisfying[4]

$$\epsilon(p,\lambda)\cdot\epsilon^*(p,\lambda') = \delta_{\lambda,\lambda'}, \qquad p\cdot\epsilon(p,\lambda) = 0, \qquad (1,0,0,0)\cdot\epsilon(p,\lambda) = 0. \tag{22}$$

There are lots of interaction vertices in the SM, and they are straightforward to work out from the Lagrangian. We'll collect only the fermion-vector couplings here. Comparing to the general notation in Fig. 1, the fermion-photon vertex for $\psi \to A_\mu\psi$ is

$$V^\mu = -ieQ\gamma^\mu. \tag{23}$$

[4]This choice isn't unique.

For the gluon, we have for $\psi_j \to G^a_\mu \psi_i$ (with i, j being the indices of the $SU(3)_c$ rep of the fermion)

$$V^\mu = -ig_s \gamma^\mu t^a_{ij}, \tag{24}$$

where t^a is the representation matrix corresponding to the $SU(3)_c$ rep of the fermion ψ_i. For a trivial representation we have $t^a = 0$ and the vertex vanishes. In the case of the Z^0, we have for $\psi \to Z_\mu \psi$

$$V^\mu = -i\bar{g}\gamma^\mu \left[(t^3 - Qs^2_W)P_L + (0 - Qs^2_W)P_R\right]. \tag{25}$$

The fermion projectors here take into account the different couplings of the Z^0 to the left- and right-handed components of the 4-component fermions we are working with. For the $W^\pm$, we have for $\psi_B \to W^-_\mu \psi'_A$ (where A, B are the flavour indices of the fermion)

$$V^\mu = -i\frac{g}{\sqrt{2}}\gamma^\mu P_L V^{(CKM)}_{AB}. \tag{26}$$

Note that here ψ is the lower component of an $SU(2)_L$ doublet while ψ' is an upper component. For $\psi'_A \to W^+_\mu \psi_B$ one gets the same vertex but with $V^{(CKM)\dagger}_{AB}$. Note also that except for the gluon coupling, the incoming and outgoing colour states at a vector vertex are the same (so we could have put on colour indices on all the fields with a δ_{ij} in the vertex). Similarly, the flavours of the incoming and outgoing fermions are identical except for the $W^\pm$ couplings and so we have not included flavour indices in the other vertices. The vertex for a fermion ψ coupling to the Higgs boson is

$$V = -i\frac{m_\psi}{\sqrt{2}v}, \tag{27}$$

where m_ψ is the fermion mass. This coupling is also diagonal in colour and flavour space.

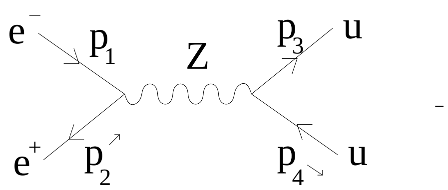
e.g. 1.) Amplitude for $e^+e^- \to u\bar{u}$ via the Z^0.
The Feynman diagram for this process is shown below. Following the rules above, we find the amplitude

$$\begin{aligned} -i\mathcal{M} \;&=\; -i\bar{g}^2\left(\frac{1}{p^2 - m_Z^2}\right)(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2) \\ &\quad \bar{u}_3\gamma^\mu\left[(\frac{1}{2} - \frac{2}{3}s^2_W)P_L + (0 - \frac{2}{3}s^2_W)P_R\right]v_4 \\ &\quad \bar{v}_2\gamma^\nu\left[(\frac{1}{2} + s^2_W)P_L + (0 + s^2_W)P_R\right]u_1. \end{aligned} \tag{28}$$

Here, $p = (p_1 + p_2) = (p_3 + p_4)$, and the subscripts label the momenta of the spinors (spinor indices are contracted). Squaring and summing/averaging this amplitude goes through very much like the $e^+e^- \to \mu^+\mu^-$ example we considered earlier for QED. The key difference is that we almost always want to sum over the colour states of the outgoing quarks. Since u and $\bar{u}$ must carry the same colour index (seeing as the Z^0 coupling does not modify colour), this introduces an additional factor of three for the three outgoing colour states. Note also that the full amplitude for $e^+e^- \to u\bar{u}$ also gets a contribution from an intermediate photon. When $|p^2| \ll m_Z^2$ the photon contribution dominates by a factor of nearly $m_Z^2/|p^2|$. Indeed, in this case the Z^0 contribution is approximated well by simply replacing the propagator by

$$(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2)/(p^2 - m_Z^2) \to \eta_{\mu\nu}/m_Z^2 \tag{29}$$

This is the form of the vertex one would optain from a point-like interaction coupling four fermions at once.



e.g. 2.) $W^+ \to u\bar{d}$.
The amplitude for this process is $(p_1 \to p_2 + p_3)$

$$-i\mathcal{M} = -i\frac{g}{\sqrt{2}}\bar{u}_2\gamma^\mu P_L V_{ud}^{(CKM)} v_3\, \epsilon_\mu(p,\lambda). \tag{30}$$

To get the physical unpolarized rate, we should average over initial states and sum over final ones. This gives

$$\begin{aligned} \text{“}|\mathcal{M}|^2\text{''} &= \frac{1}{3}\sum_i\sum_\lambda\sum_{s,s'}|\mathcal{M}|^2 \qquad (31) \\ &= \frac{g^2}{2}|V_{ud}|^2\left[2(p_2^\mu p_3^\alpha + p_2^\mu p_3^\alpha - p_2\cdot p_3\eta^{\mu\alpha}) - 2i\epsilon^{\rho\mu\sigma\alpha}p_{2_\rho}p_{3_\sigma}\right](-\eta_{\mu\alpha} + p_{1_\mu}p_{1_\alpha}/m_W^2) \\ &\simeq g^2|V_{ud}|^2 m_W^2. \end{aligned}$$

In the first line, the sums run over the colours of the quarks, the polarizations of the initial W^+ (with a 1/3 factor to make it into an average over initial pols), and a sum over final state spins. In the last line we've ignored the u and d masses which are much smaller than the W mass and correct this result by factors of $m_{u,d}^2/m_W^2$. In this approximation, the partial decay width for this channel is

$$\Gamma(W^+ \to u\bar{d}) = \frac{g^2}{8\pi}|V_{ud}|^2 m_W. \tag{32}$$

This is typical for a 2-body decay width - it goes like $(mass)(coupling)^2/16\pi$ up to factors of order unity.

References

[1] ...

[2] See Appendix A in:
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