PHYS 528 Lecture Notes #1

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1 Notational Conventions

In elementary particle physics, we are usually interested in highly relativistic systems. For this reason, we will work in a Lorentz-covariant language. Some of the key bits of notation related to this are:

$$
\begin{align}
x^\mu &= (t, x, y, z), \quad \mu = 0, 1, 2, 3 \tag{1}\\
p^\mu &= (E, p^x, p^y, p^z) \tag{2}\\
x_\mu &= \eta_{\mu\nu} x^\nu, \qquad p_\mu = \eta_{\mu\nu} p^\nu \tag{3}\\
\eta_{\mu\nu} &= diag(+1, -1, -1, -1), \quad \eta^{\mu\nu} = diag(+1, -1, -1, -1) \tag{4}\\
\epsilon^{\mu\nu\lambda\kappa} &= \text{completely antisymmetric with } \epsilon^{0123} = +1. \tag{5}
\end{align}
$$

Note that the convention I'll use for the metric is opposite to that used in the main text, Burgess&Moore[1]. Sorry about this, but my convention is by far the more commonly-used one in particle physics, and it's a useful skill to know how to convert from one convention to another.

Here, whenever an index is repeated, it is implicitly summed over. Indices are raised and lowered with the matrix $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. For instance,

$$
\begin{align}
x^2 &\equiv x^\mu x_\mu = \eta_{\mu\nu} \, x^\mu x^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} \, x^\mu x^\nu = t^2 - \vec{x}^2, \tag{6}\\
p_\mu &= \eta_{\mu\nu} p^\nu. \tag{7}
\end{align}
$$

The general rule-of-thumb with this notation is that a quantity is Lorentz-invariant iff all the indices are summed over. Thus, x^2 and p^2 are Lorentz-invariant, while x_μ and p^μ are not. We'll also write derivatives as

$$
\begin{align}
\partial_\mu &= \frac{\partial}{\partial x_\mu} \tag{8}\\
\partial^2 &= \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_t^2 - \vec{\nabla}^2 \tag{9}
\end{align}
$$

In passing, let me add that the convention of summing over repeated indices is sometimes called the *Einstein summation convention*.[1]

We will also work in *natural units*, with

$$
\hbar = c = 1. \tag{10}
$$

---

[1] Apparently he felt that it was his greatest contribution to physics [7].

This simplifies dimensional analysis since now all dimensionful quantities can be expressed in units of energy. For example,

$$[E] = [P] = [M] = +1, \quad [L] = [T] = -1, \quad [\partial_\mu] = +1, \tag{11}$$

where the square brackets denotes the energy dimension of the quantity (in natural units, of course). To put a result back in regular units, just stick in powers of $\hbar$ ($\sim ET$) and c ($\sim L/T$) until you get what you want. Three handy things to remember are $\hbar c = 1 \simeq 0.197 GeV\, fm$, $c = 1 \simeq 3 \times 10^{10} cm^2/s$, and $1\, pb = 10^{-36} cm^2$.[2]

When dealing with fermions, we will also make use of Pauli and Dirac matrices. For the Pauli matrices, we will write

$$\sigma^0 = \mathbb{I}, \quad \sigma^i = \sigma^{1,2,3} \tag{12}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

Recall that

$$\sigma^i \sigma^j = \delta^{ij} \mathbb{I} + i\epsilon^{ijk} \sigma^k. \tag{14}$$

Let us also define

$$\sigma^\mu = (\mathbb{I}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma}). \tag{15}$$

The Dirac matrices (in any number of spacetime dimensions) satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{16}$$

In four spacetime dimensions, they can be written as 4×4 matrices. A useful choice is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{17}$$

This choice is sometimes called the *chiral representation*. We will also define

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\lambda\kappa}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\kappa. \tag{18}$$

In the chiral representation, one finds

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{19}$$

We will also encounter the chiral projectors $P_L = (1 - \gamma^5)/2$ and $P_R = (1 + \gamma^5/2)$.

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[2]A pb is a unit commonly used to express scattering cross-sections. It is 10^{-12} barns, with $1b = 10^{-24} cm^2$.

2 Quick Review of Quantum Field Theory

Quantum Field Theory (QFT) is the best tool we know of to describe the physics of elementary particles. It is the basic language used in elementary particle physics, superstring theory, and many branches of condensed matter physics. Learning QFT has the reputation of being difficult, but it is more accurate to say that it is time-consuming. Since I can't cram a whole year's worth of material into a single class, I will try to give you a general flavour of the topic. For a list of useful textbooks, take a look at Refs. [2, 3, 4, 5, 6]. I also strongly encourage you to take a full QFT course at some point in your graduate careers.

QFT is nothing more than ordinary quantum mechanics formulated in a relativistically invariant way and applied to continuous *field* systems. The only difference from the more-familiar one-particle quantum mechanics is what we identify as the underlying degrees of freedom, which in this case are the fields. It might seem strange to use continuous fields to describe discrete objects like particles. When a field is quantized, however, there appear discrete quantum excitations that we interpret as particles. This interpretation is justified *a posteriori* by its excellent agreement with experiment. On the other hand, ordinary one-particle quantum mechanics doesn't get along well with (special) relativity because it doesn't account for particle creation at high energies. Quantum field theory avoids this problem, and reduces to one-particle quantum mechanics in the appropriate limit. For a nice alternative point of view on why we use QFT to describe elementary particles, read Ch.1 of Burgess&Moore [1].

We usually define a QFT in terms of an *action* that depends on a set of fields $\{\phi_i\}$. For the cases of interest to us, the action can be written in the form

$$S[\phi_i] = \int d^4x \, \mathscr{L}(\phi_i(x)). \tag{20}$$

The function $\mathscr{L}(\phi_i)$ is the Lagrange density (but we'll call it the Lagrangian). Some comments about the action:

- S depends on x^μ only through the fields $\phi_i(x)$. This implies invariance under spacetime translations.

- S should also be invariant under Lorentz boosts and rotations. These take the form

$$
\begin{align}
x^\mu &\to \Lambda^\mu{}_\nu x^\nu := x'^\mu(x) \tag{21} \\
\phi_\alpha(x) &\to U_\alpha{}^\beta \phi_\beta(x'(x)) := \phi'_\alpha(x) \tag{22}
\end{align}
$$

  Since $d^4x' = d^4x$, Lorentz invariance requires that $\mathscr{L}(\phi) \to \mathscr{L}'(\phi') = \mathscr{L}(\phi')$ (up to total derivatives). In other words, the Lagrangian as a function of the transformed has the same functional form as the original Lagrangian.

- The action is *local*, in that it only depends on functions (and derivatives) fields at the same spacetime point. (*e.g.* There are no terms like $\int d^4x \int d^4y \, \phi(x)\phi(y)$ in the action.) This is needed if we want our theory to be causal.

3

- Dimensions: $[S] = 0$, $[d^4x] = -4$, so we need $[\mathscr{L}] = +4$.

- S needs to be real for the theory to be unitary.

An action that satisfies these conditions can potentially give rise to a well-defined, Lorentz-invariant QFT. Because of Lorentz Invariance (LI), we can work with fields that have well-defined transformation properties under Lorentz transformations. We identify these fields with particles of definite spins:

Scalar $(\mathbf{s = 0})$: $\phi'(x) = \phi(x')$
Fermion $(\mathbf{s = 1/2})$: $\psi'(x) = U_a{}^b \psi_b(x')$ (The indices here are *spinor* indices.)
Vector $(\mathbf{s = 1'})$: $A'_\mu(x) = \Lambda_\mu{}^\nu A_\nu(x')$

A handy rule of thumb is that the Lagrangian will be LI provided all the indices on the fields are properly contracted.

In practice, we want to use QFT to compute things that can be compared to experiment. The standard technique for this is to make a perturbative expansion of the QFT around the non-interacting free-field theory consisting only of quadratic (and lower) powers of the fields.

Rules:

1. Start with the quadratic (and lower) terms in the Lagrangian and extract from them the kinetic and mass terms.

2. For this, redefine the field variables to put the kinetic terms in *canonical* form and diagonalize the mass matrices.

3. Add the terms higher than quadratic (in terms of the redefined and now-canonical/diagonal fields) and compute perturbatively with Feynman diagrams.

e.g. 1.a) Real Scalar Fields; ϕ_i, $i = 1, 2, \ldots n$ label different scalars.

$$\mathscr{L} = C^i \phi_i + \frac{1}{2} Z^{ij} \eta^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_j - \frac{1}{2}(M^2)^{ij} \phi_i \phi_j + (H.O.T.) \tag{23}$$

Here, C^i is a constant while Z^{ij} and $(M^2)ij$ are real, strictly positive-definite, symmetric matrices.
Step 1: remove the linear term with $\phi_i = \phi'_i + \alpha_i$ with α_i a constant.
In terms of the new field variables,

$$\mathscr{L} = (const) + (C^i - (M^2)^{ij}\alpha_j)\phi'_i + \frac{1}{2} Z^{ij} \partial\phi'_i \partial\phi'_j - \frac{1}{2}(M^2)^{ij}\phi'_i\phi'_j + (H.O.T.) \tag{24}$$

By choosing the α_i judiciously, we can remove the linear term. We don't care about the constant term that is generated.

Step 2: make the kinetic term "canonical".

For this, rotate the fields by $\phi_i' = P_i^j \phi_j''$. We can choose P_i^j to be a product of an orthogonal matrix times a diagonal matrix. The orthogonal matrix should diagonalize $Z^{ij} \to diag(z_1, z_2, \ldots z_n)$ while the diagonal matrix is $diag(z_1^{-1}, z_2^{-1}, \ldots z_n^{-1})$. Note that for the field theory to be sensible, there should be no negative or zero eigenvalues. This produces

$$\mathscr{L} = (const) + \frac{1}{2}\delta^{ij}\partial\phi_i''\partial\phi_j'' - \frac{1}{2}\tilde{M}^2 ij\phi_i''\phi_j'' + (H.O.T.) \tag{25}$$

Here, $(\tilde{M}^2)^{ij} = P_k^i (M^2)^{kl} P_l^j$.

Step 3: diagonalize the mass matrix.
To do so, make an orthogonal transformation $\phi_i'' = \mathcal{O}_i^j \phi_j'''$. This is always possible because the mass matrix is real symmetric. Note that this won't mess up the canonical form of the kinetic term. At the end of the day, we get

$$\mathscr{L} = (const) + \frac{1}{2}\sum_i [(\partial\phi_i)^2 - m_i^2\phi_i^2] + (H.O.T.) \tag{26}$$

The $(H.O.T.)$ stuff will now be a mess, but this is just something we'll have to deal with.

e.g. 1.b) Complex Scalar Field

The canonical form is

$$\mathscr{L} = \eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2|\phi|^2. \tag{27}$$

Note that we can also write this in terms of two real scalar fields, $\phi = (\phi_R + i\phi_I)/\sqrt{2}$.

e.g. 2. Dirac Fermion Field

The canonical form is

$$\mathscr{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi + (H.O.T.) \tag{28}$$

e.g. 3. Vector Field

The canonical form is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu + (H.O.T.) \tag{29}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The signs of the two terms here might look a bit funny, but they're related to the fact that A^0 is not entirely independent. Instead, it can be related to the other components by the constraint $\partial_\mu A^\mu$. Incorporating this requirement leads to the funny-looking kinetic term. The sign of the mass term is sensible once you remember that $A_\mu A^\nu = (A^0)^2 - \vec{A}\cdot\vec{A}$. (See B&M for more details.)

3 Quantum Electrodynamics (QED)

This is the theory of charged fermions interacting with the photon. It consists of a massless vector for the photon and a set of charged fermions. The Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_i \bar{\psi}_i[i\gamma^\mu(\partial_\mu + ieQ_iA_\mu) - m_i]\psi_i \tag{30}$$

Sometimes we'll write $D_\mu = (\partial_\mu + ieQ_iA_\mu)$.

From this one can derive the following Feynman rules: Here, $u(p,s)$ and $v(p,s)$ are 4×1

Incoming Fermion	$s \xrightarrow{p} \bullet$	$u(p,s)$
Incoming Anti–Ferm	$s \xrightarrow{p}\bullet$	$\bar{v}(p,s)$
Outgoing Fermion	$\bullet \xrightarrow{p} s$	$\bar{u}(p,s)$
Outgoing Anti–Ferm	$\bullet \xrightarrow{p} s$	$v(p,s)$
Incoming Photon	$\mu,\lambda \sim\!\!\sim\!\!\sim^{p}\bullet$	$\epsilon_\mu(p,\lambda)$
Outgoing Photon	$\bullet\sim\!\!\sim\!\!\sim^{p}_{\mu,\lambda}$	$\epsilon_\mu^*(p,\lambda)$
Internal Fermion	$\bullet \xrightarrow{p} \bullet$	$i(\not{p} + m)/(p^2 - m^2)$
Internal Photon	$\mu\bullet\sim\!\!\sim\!\!\sim^{p}\bullet\nu$	$-i\eta_{\mu\nu}/p^2$
Vertex	$\mu\sim\!\!\sim\!\!\sim\!\!<$	$-ieQ\gamma^\mu$

fermion and anti-fermion spin vectors for momentum p and spin s, with $\bar{u} = u^\dagger\gamma^0$. We have also written $\not{p} = p_\mu\gamma^\mu$. For photons, $\epsilon_\mu(p,\lambda)$ is the polarization vector for the polarization state λ. Recall that photons have two independent transverse polarizations.

To compute a quantum mechanical scattering amplitude:

1. Draw all possible Feynman diagrams at the desired level of perturbative approximation. (We'll stick to the leading order here.)

2. For each diagram, follow the rules listed above to find the mathematical expression for each diagram. For closed loops, integrate over internal loop momentum and add a factor of (-1) if it is a fermion loop.

3. Sum the expressions for each of the diagrams. Include any necessary symmetry factors for identical initial or final states. In the case of fermions, diagrams that differ only by the interchange of two identical fermion lines come in with a relative factor of (-1).

4. The result of all this is $(-i)$ times the amplitude.

e.g. 1.a) Amplitude for $e^+e^- \to \mu^+\mu-$

The Feynman diagram for this process is shown below. Following the rules above, we find the amplitude

$$-i\mathcal{M} = ie^2 Q_e Q_\mu \frac{1}{p^2}(\bar{u}_3 \gamma^\mu v_4)(\bar{v}_2 \gamma^\nu u_1)\eta_{\mu\nu}. \tag{31}$$

Here, $p = (p_1 + p_2) = (p_3 + p_4)$, and the subscripts label the momenta of the spinors (spinor indices are contracted).
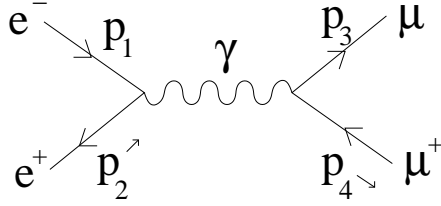
Once we have the amplitude, all we need to do is specify the initial and final fermion spin or photon polarization states and square the result to find the squared matrix element into a scattering cross section or a decay rate. However, in many cases we are interested only in the inclusive unpolarized cross-section. For this, we should *average* over initial spin/polarization states and sum over final spin/polarization states. There are a number of tricks for doing this that make use of the completeness properties of the fermion spinors and the photon polarization vectors.

Spinor Tricks:

$$
\begin{aligned}
(\not{p} - m)\, u(p,s) &= 0 = (\not{p} + m)\, v(p,s) & (32) \\
\sum_s u_a(p,s)\bar{u}_b(p,s) &= (\not{p} + m)_{ab} & (33) \\
\sum_s v_a(p,s)\bar{v}_b(p,s) &= (\not{p} - m)_{ab} & (34) \\
\gamma^0(\gamma^\mu)^\dagger\gamma^0 &= \gamma^\mu & (35) \\
tr(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} & (36) \\
tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa) &= 4(\eta^{\mu\nu}\eta^{\lambda\kappa} + \eta^{\mu\kappa}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\kappa}) & (37) \\
tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa\gamma^5) &= -4i\epsilon^{\mu\nu\lambda\kappa} & (38) \\
\{\gamma^\mu\gamma^\nu\} &= 2\eta^{\mu\nu} & (39) \\
\{\gamma^5, \gamma^\mu\} &= 0 & (40)
\end{aligned}
$$

Additional tricks can be found in Ref. [5]. Note that the subscripts in Eqs.(33,34) are spinor indices.

Photon Tricks:

$$\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + \text{(stuff you can ignore)}. \tag{41}$$

e.g. 1.b) Squared and summed amplitude for $e^+ e^- \to \mu^+ \mu-$
What we want is

$$
\begin{aligned}
\text{``}|\mathcal{M}|^{2}\text{''} &= \frac{1}{2} \times \frac{1}{2} \times \sum_{s_1, s_2, s_3, s_4} |\mathcal{M}(s_1 s_2 \to s_3 s_4)|^2 \qquad (42) \\
&= \frac{1}{4}(e^2 Q_e Q_\mu \frac{1}{p^2})^2 \sum_{s...} \eta_{\mu\nu}(\bar{u}_3 \gamma^\mu v_4)(\bar{v}_2 \gamma^\nu u_1)[\eta_{\alpha\beta}(\bar{u}_3 \gamma^\alpha v_4)(\bar{v}_2 \gamma^\beta u_1)]^\dagger.
\end{aligned}
$$

Let's first conjugate the 34 spinor piece. We have

$$
\begin{aligned}
(\bar{u}_3 \gamma^\alpha v_4)^\dagger &= v_4^\dagger (\gamma^\alpha)^\dagger \gamma^0 (u_3^\dagger)^\dagger \qquad (43) \\
&= v_4^\dagger \gamma^0 \gamma^0 (\gamma^\alpha)^\dagger \gamma^0 u_3 \\
&= \bar{v}_4 \gamma^\alpha u_3.
\end{aligned}
$$

The 12 piece goes through similarly. Next, we can assemble the 12 and 34 pieces and use the spinor completeness relations. For the 34 part, we get

$$
\sum_{s_3, s_4} \bar{u}_3 \gamma^\mu v_4 \, \bar{v}_4 \gamma^\alpha u_3 \qquad (44)
$$

$$
\begin{aligned}
&= \sum_{s_3} \sum_{s_4} \bar{u}_{3a} \gamma^\mu_{ab} v_{4b} \bar{v}_{4c} \gamma^\alpha_{cd} u_{3d} \qquad (45) \\
&= (\not{p} + m_3)_{da} \gamma^\mu_{ab} (\not{p} - m_4)_{bc} \gamma^\alpha_{cd} \\
&= tr[(\not{p} + m_3) \gamma^\mu (\not{p} - m_4) \gamma^\alpha] \\
&= 4(p_3^\mu p_4^\alpha + p_3^\alpha p_4^\mu - p_3 \cdot p_4 \eta^{\mu\alpha}) - 4 m_3 m_4 \eta^{\mu\alpha}. \qquad (46)
\end{aligned}
$$

I've written out the spinor indices in gory detail here, but you can skip this part once you get the hang of it. Combining with the 12 piece and contracting indices, the result (in the limit $p^2 \gg m_{e,\mu}^2$ such that we can neglect the masses) is

$$
\text{``}|\mathcal{M}|^{2}\text{''} = 8(e^2 Q_e Q_\mu \frac{1}{p^2})^2 [(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]. \qquad (47)
$$

Working in the centre-of-mass (CM) frame, we have (after applying energy and momentum conservation)

$$
\begin{aligned}
p_1 &= (q, 0, 0, q), & p_2 &= (q, 0, 0, -q) \qquad (48) \\
p_3 &= (q, q\sin\theta, 0, q\cos\theta), & p_4 &= (q, -q\sin\theta, 0, -q\cos\theta) \qquad (49)
\end{aligned}
$$

The summed and squared matrix element is then

$$
\text{``}|\mathcal{M}|^{2}\text{''} = 16 e^4 Q_e^2 Q_\mu^2 (1 + \cos^2 \theta). \qquad (50)
$$

The two observable quantities we are most interested in calculating are scattering cross sections and decay rates.

Cross Sections $(2 \to n)$

$$d\sigma(a + b \to 1 + \ldots + n) = \frac{1}{|v_a - v_b|} \frac{1}{4E_a E_b} (d\tilde{p}_1) \ldots (d\tilde{p}_n) \, (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_i p_i) \, "|\mathcal{M}|^{2"} \quad (51)$$

where

$$d\tilde{p}_i = \frac{d^3 p_i}{(2\pi)^3 E_i} \quad (52)$$

Decay Rates $(1 \to n)$

$$d\Gamma(a \to 1 + \ldots + n) = \frac{1}{2E_a} (d\tilde{p}_1) \ldots (d\tilde{p}_n) \, (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_i p_i) \, "|\mathcal{M}|^{2"} \quad (53)$$

References

[1] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[2] For an informal and very intuitive introduction to QFT, take a look at:
A. Zee, "Quantum field theory in a nutshell," *Princeton, UK: Princeton Univ. Pr. (2003) 518 p*

[3] A new QFT textbook that looks good (but that I am not very familiar with) is:
M. Srednicki, *Cambridge, UK: Univ. Pr. (2007) 641 p*

[4] This book gives a non-standard but very clever introduction to QFT, and some basic string theory as well. There are a number of topics in this book that can be useful but are rarely covered in other texts, such as the Schr odinger representation of QFT in terms of *wave functionals*.
B. Hatfield, "Quantum field theory of point particles and strings," *Redwood City, USA: Addison-Wesley (1992) 734 p. (Frontiers in physics, 75)*

[5] This is the standard modern QFT textbook. It is probably the most complete single text on the subject with excellent exercises. If you want to learn QFT and have to choose a single book, this is it. However, because it is so complete, some specific topics are treated in more detail elsewhere.
M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[6] L. H. Ryder, "Quantum Field Theory," *Cambridge, Uk: Univ. Pr. (1985) 443p*

[7] H. Goldstein, "Classical Mechanics", *Cambridge USA: Addison-Wesley (1950)*