PHYS 526 Notes #8: Interacting Fermions

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We turn next to study theories with both fermions and interactions.

1 Perturbation Theory

As a specific example, we will study the interacting field theory of a real scalar ϕ and a Dirac fermion Ψ given by

$$\mathscr{L} = \bar{\Psi}i\partial\!\!\!/\Psi - m\bar{\Psi}\Psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - y\,\phi\bar{\Psi}\Psi \ . \tag{1}$$

The last term is evidently the interaction piece, $\Delta\mathscr{H} = \Delta V = y\phi\bar{\Psi}\Psi$. It is called a *Yukawa interaction* (and y is called the Yukawa coupling) after H. Yukawa who first suggested this form to describe the strong interaction between two nucleons (fermions) and a pion (scalar).

1.1 The Master Formula

We already know all the energy eigenstates of the theory in the limit of $y \to 0$. They are just free particle states containing some number of scalars, fermions, and antifermions, all with definite momentum (and spin for the fermions). However, as soon as we turn on the coupling, life becomes much more complicated. Even so, we can still study the interacting theory as an expansion in the coupling around the free theory. For this, we make two assumptions:

1. There exists a unique vacuum $|\Omega\rangle$ of the full theory with $p^\mu|\Omega\rangle = 0$.
2. For each field in the Lagrangian, there exists a set of distinct one-particle momentum eigenstates (possibly with several spin/helicity sub-states). Such states correspond to isolated poles in the two-point functions of the elementary fields.

Note that these are both assumptions. They turn out to be valid in many theories with small couplings, but they are also known to be broken in theories with large couplings. Not surprisingly, pertubation theory does not work very well at all at strong coupling, and the physical states might not correspond to fields in the Lagrangian (or even be particle-like).

The first things we will attempt to compute are the n-point functions, the expectation values of time-ordered products of fields. For this, we use the obvious generalization of the master formula:

$$\langle\Omega|T\left\{\mathcal{O}(x,y,z)\right\}|\Omega\rangle = \frac{\langle 0|T\left\{\mathcal{O}_I(x,y,z)\exp\left[-i\int d^4w\;\Delta\mathscr{H}_I(w)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int d^4w\;\Delta\mathscr{H}_I(w)\right]\right\}|0\rangle}\ , \tag{2}$$

where $\mathcal{O}(x,y,z)$ is a shorthand for

$$\mathcal{O}(x,y,z) := \phi(x_1)\ldots\phi(x_\ell)\ \Psi_{a_1}(y_1)\ldots\Psi_{a_m}(y_m)\ \bar{\Psi}^{b_1}(z_1)\ldots\bar{\Psi}^{b_n}(z_n)\ . \tag{3}$$

Note that the quantities on the left-hand side of Eq. (2) refer to the vacuum and the Heisenberg-picture fields of the full theory, while the quantities on the right-hand side involve the vacuum of H_0 and the interaction-picture fields. Furthermore, the endpoints of the time integration within the exponentials should formally be viewed as the limits of $w^0 \to \pm\infty(1-i\epsilon)$. All these features are identical to the bosonic case, and the derivation goes through in the same way. The only substantial difference is that the time ordering picks up an extra minus sign whenever two fermion fields are interchanged.

1.2 Wick's Theorem

We apply the master formula of Eq. (2) to compute n-point functions by expanding the exponenentials to a fixed order in the couplings and evaluating the vacuum expectation value of the resulting products of fields. A useful result for doing so is Wick's theorem as generalized to fermions. The result is (for free fields but also applicable to interaction-picture fields)

$$T\{\mathcal{O}(x,y,z)\} = N\{\mathcal{O}(x,y,z) + \text{all contractions}\}\ . \tag{4}$$

This is identical to what we had for the scalar theory, although we must still define what we mean by a contraction. The correct definitions are:

$$\overbracket{\phi(x)\phi}(x') = D_F(x-x') \tag{5}$$

$$\overbracket{\Psi_a(x)\bar{\Psi}^b}(x') = [S_F(x-x')]_a^{\ b} = -\,\overbracket{\bar{\Psi}^b(x')\Psi_a}(x) \tag{6}$$

$$\overbracket{\Psi(x)\Psi}(x') = 0 = \overbracket{\bar{\Psi}(x)\bar{\Psi}}(x') \tag{7}$$

$$\overbracket{\phi(x)\Psi}(x') = 0 = \overbracket{\phi(x)\bar{\Psi}}(x')\ . \tag{8}$$

These contractions reflect the structure of the 2-point functions in the free theory, and should not be too surprising. The only thing to remember is to add a minus sign for every time you anticommute a pair of fermionic fields.

1.3 Feynman Rules

We are now ready to formulate Feynman rules for n-point functions in perturbation theory. It is obvious from our generalization of Wick's theorem that the 2-point functions at leading order are identical to those of the free theory. Beyond this, we must deduce a vertex factor, and also figure out how to handle multiple fermion lines. As before, we will begin by assigning a Feynman diagram to each distinct contraction contributing to the n-point function. Once we've seen how this works, we will go in the other direction and spell out how to compute the contractions using Feynman diagrams alone.

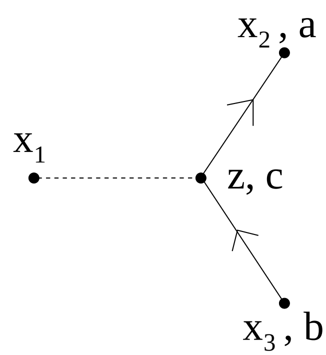

Figure 1: Feynman diagram for the $\phi\Psi\bar{\Psi}$ 3-point function at leading order.

Let us compute the 3-point function $\phi(x_1)\Psi_a(x_2)\bar{\Psi}^b(x_3)$ to leading order in the coupling y. We have a single, unique contraction:

$$\langle\phi_1\Psi_{2\,a}\bar{\Psi}_3^b\rangle = \langle 0|T\{\phi_1\Psi_{2\,a}\bar{\Psi}_3^b\,(-iy)\int d^4z\;\phi_z\bar{\Psi}_z^c\Psi_{z\,c}\}|0\rangle \tag{9}$$

$$= (-iy)\int d^4z\;D_F(x_1-z)\,[S_F(x_2-z)]_a^{\;\;c}\,[S_F(z-x_3)]_c^{\;\;b}\;. \tag{10}$$

To do the fermion contractions, $\bar{\Psi}_3^b$ can be moved first all the way to the right, giving a factor of $(-1)^2$, and then everything can be connected up.

The Feynman diagram that we assign to this result is shown in Fig. 1. The dashed line denotes the scalar propagator factor D_F, and the solid lines correspond to the fermion propagators S_F. We have also assigned arrows to the fermion lines to show the direction of "index flow", since $[S_F]_a^{\;\;b}$ is a matrix connecting the index a to the index b. With our choice of conventions, this matches up conveniently with the flow of fermion number. By following the lines backwards, we automatically pick up the correct Dirac index structure, with a connecting to c and c connecting to b. Note as well that the internal index c gets summed over.

With an eye on scattering, let us take the Fourier transform of this result,

$$\int d^4x_1\,e^{-ip_1\cdot x_1}\int d^4x_2\,e^{ip_2\cdot x_2}\int d^4x_3\,e^{-ip_3\cdot x_3}\;\langle\phi_1\Psi_{2\,a}\bar{\Psi}_3^b\rangle$$
$$= (-iy)\,(2\pi)^4\delta^{(4)}(p_2-p_1-p_3)\,\frac{i}{p_1^2-M^2}\left[\frac{i(\not{p}_2+m)}{p_2^2-m^2}\frac{i(\not{p}_3+m)}{p_3^2-m^2}\right]_a^{\;\;b}\;. \tag{11}$$

You can probably already see how the momentum-space Feynman rules are going to turn out. As before, the Dirac indices contract by following the fermion lines backwards. Note that I have also chosen an opposite sign in the exponential for p_2 relative to the others. This choice corresponds to p_2 flowing out, while p_1 and p_3 both flow in. It is convenient in this case because the momentum flow matches up with the fermion number flow. This doesn't always have to be the case.

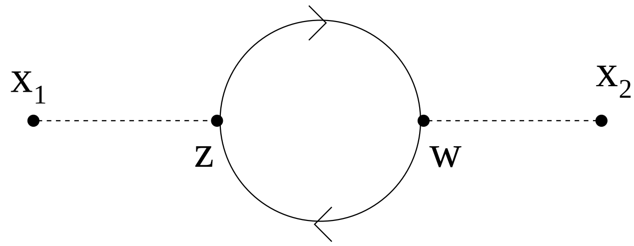

Figure 2: Feynman diagram for the connected part of the $\phi\phi$ 2-point function at y^2 order.

As a second example, consider the connected y^2 corrections to the scalar 2-point function. These give

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle & \\ = & -(-iy)^2\int d^4z\int d^4w\; D_F(z-x_1)[S_F(w-z)]_a{}^b[S_F(z-w)]_b{}^a D_F(x_2-w) \quad (12)\\ = & -(-iy)^2\int d^4z\int d^4w\; D_F(z-x_1)D_F(x_2-w)\,tr[S_F(z-w)S_F(w-z)]\;. \quad (13)\end{aligned}$$

The corresponding Feynman diagram is shown in Fig. 2. We see that the fermion loop produces a trace over the Dirac indices. This loop also generates a factor of (-1) from the fermion rearrangements needed to contract them all. Both features always occur for closed fermion loops. In momentum space, the result is

$$\begin{aligned}&\int d^4x_1\, e^{-ip_1\cdot x_1}\int d^4x_2\, e^{ip_2\cdot x_2}\;\langle\phi_1\phi_2\rangle \quad (14)\\ &= -(-iy)^2\left(\frac{i}{p_1^2-M^2}\right)^2\int\frac{d^4q}{(2\pi)^4}\,tr\left[\frac{i(\not{p}_1+\not{q}+m)}{(p_1+q)^2-m^2}\frac{i(\not{q}+m)}{q^2-m^2}\right]\,(2\pi)^4\delta^{(4)}(p_1-p_2)\;.\end{aligned}$$

Using the gamma matrix tricks discussed in `notes-07`, it is straightforward to evaluate the trace and reduce it to a function of q, p_1, and m.

Based on these results, we define the $(\ell+m+n)$-point function ($\ell\phi$ fields, $m\Psi$ fields, $n\bar{\Psi}$ fields) in momentum space exactly as before:

$$(2\pi)^4\delta^{(4)}\left(\sum_{i=\ell,m,n}p_i\right)\tilde{G}^{(\ell+m+n)}(\{p\}) = \left(\prod_{i=\ell,m,n}\int d^4x_i e^{-ip_i\cdot x_i}\right)G^{(\ell+m+n)}(\{x\})\;, \quad (15)$$

where $G^{(\ell+m+n)}(\{x\})$ is the corresponding quantity in position space obtained from the master formula. While we know how to do this, it can be very tedious. Instead, it is more efficient to calculate $\tilde{G}(\{p\})$ directly using a set of (momentum space) Feynman rules:

1. Draw an external line for each momentum p_i with one free end and one fixed end. Also, for each fermion line corresponding to $\Psi_a(y)$, assign the Dirac index a to the external point and draw an arrow on the line directed at the point. Similarly, for each line corresponding to $\bar{\Psi}^b(z)$ include the Dirac index b at the external point and draw an arrow on the line directed away from the point.

2. Draw another M dots corresponding to the vertices, and assign a Dirac index c to each one. Each dot should have one scalar line attached to it, one fermion line directed out of it, and one fermion line directed into it.

3. Assemble all possible Feynman diagrams by connecting up the free ends of lines in pairs in all possible ways subject to two rules:

   a) Scalar lines can only connect to scalar lines.

   b) Fermion lines can only be connected if their arrows point in the same direction.

   Also, do not include diagrams that are related by permuting the labels of the vertices. (The $M!$ such permutations cancel the $1/M!$ from expanding the exponential.)

4. Remove all diagrams containing vacuum bubbles and any diagrams with one or more unconnected free ends.

5. Assign a value to each diagram:

   a) Assign a momentum to every line. Each line connected to an external point gets a four momentum p_i directed inwards (away from the point). Having fixed these, constrain the momenta of all internal lines by imposing four-momentum conservation at every vertex. This may still leave a few internal momenta undertermined. Call them q_j for now.

   b) Each scalar line with momentum p gets a propagator factor of $i/(p^2 - M^2 + i\epsilon)$. Each fermion line with the momentum flowing parallel to the fermion number direction gets a factor of $i(\not{p} + m)_a{}^b/(p^2 - m^2 + i\epsilon)$, where a is the Dirac index at the tip of the line and b is Dirac index at the tail (as determined by the direction of the arrow on the line). Also, if the momentum is antiparallel to the fermion number arrow, flip the sign of the momentum in the propagator $p \to -p$ so that the line gets a factor of $i(-\not{p} + m)_a{}^b/(p^2 - m^2 + i\epsilon)$.

   c) Write a factor of $-iy$ for each vertex.

   d) Integrate over all undetermined momenta $\int d^4q_j/(2\pi)^4$ and sum over all internal Dirac indices c.

   e) For each set of external momenta p_i connected to each other in some way, multiply by an overall factor of $(2\pi)^4\delta^{(4)}\left(\sum p_i\right)$.

   f) Multiply the diagram by the symmetry factor and whatever factors of (-1) that were incurred by moving fermions around in the contractions. In general, a closed fermion loop always picks up a net factor of (-1), and any two diagrams that differ by the exchange of fermion legs have a relative sign between them.

The resulting sum of all the diagrams is the order y^M contribution to $(2\pi)^4\delta^{(4)}(\sum p)\ \tilde{G}(p)$.

To illustrate these rules in action, let us compute the two diagrams shown in Fig. 3, that occur in the y^2 contribution to $\langle \Psi_{a_1}(x_1)\bar{\Psi}_{a_2}(x_2)\Psi_{a_3}(x_3)\bar{\Psi}_{a_4}(x_4)\rangle$. For the first (on the left),

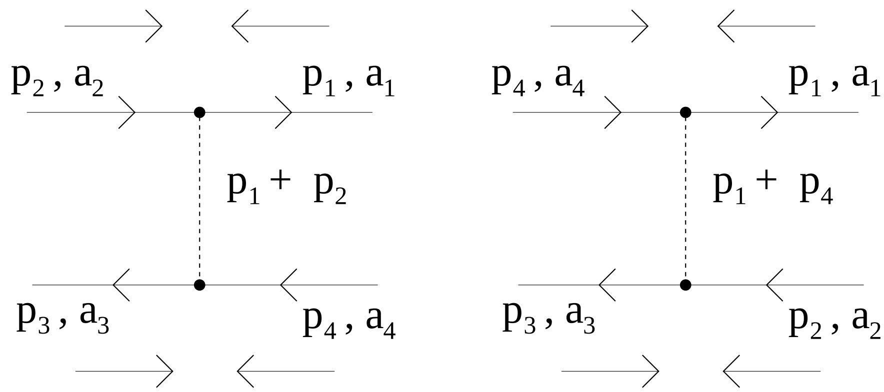

Figure 3: Feynman diagram for the connected part of the $\Psi\bar{\Psi}\Psi\bar{\Psi}$ 4-point function.

we get

$$
\begin{aligned}
D_1 &= (-iy)^2 \frac{i}{(p_1+p_2)^2 - M^2}\,(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right) \\
&\quad \times \frac{i(-\not{p}_1+m)_{a_1}{}^{c}}{p_1^2-m^2}\frac{i(\not{p}_2+m)_{c}{}^{a_2}}{p_2^2-m^2}\frac{i(-\not{p}_3+m)_{a_3}{}^{d}}{p_3^2-m^2}\frac{i(\not{p}_4+m)_{d}{}^{a_4}}{p_4^2-m^2}\,,
\end{aligned}
\tag{16}
$$

while for the second diagram (on the right) we find

$$
\begin{aligned}
D_2 &= (-1)(-iy)^2 \frac{i}{(p_1+p_4)^2 - M^2}\,(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right) \\
&\quad \times \frac{i(-\not{p}_1+m)_{a_1}{}^{c}}{p_1^2-m^2}\frac{i(\not{p}_4+m)_{c}{}^{a_4}}{p_4^2-m^2}\frac{i(-\not{p}_3+m)_{a_3}{}^{d}}{p_3^2-m^2}\frac{i(\not{p}_2+m)_{d}{}^{a_2}}{p_2^2-m^2}\,.
\end{aligned}
\tag{17}
$$

Note the relative minus sign in D_2 relative to D_1. Again, we see that the general trick to figuring out the Dirac contraction structure is to simply work backwards along the fermion lines.

2 Asymptotic States and Scattering

Our next challenge is to convert our Feynman rules for n-point functions into amplitudes for particle scattering. The procedure for this goes through just like for scalars but with a few twists. We will not go into the details, but we will present the main results.

2.1 Spectral Decomposition

The spectral decomposition for an interacting Dirac fermion is

$$
\int d^4x\, e^{-ip\cdot x}\langle \Psi_a(x)\bar{\Psi}^b(0)\rangle = \frac{iZ(\not{p}+m)_a{}^b}{p^2-m^2+i\epsilon} + \int_{>m^2}^{\infty}\frac{ds}{2\pi}\,\frac{W_a{}^b(s)}{p^2-s+i\epsilon}\,,
\tag{18}
$$

where the first term comes from our assumption about the existence of a one-particle state and ${W_a}^b$ is some unspecified function of s in the Dirac space where we dump everything else. The derivation of this result goes through just like in the scalar case. Note that the location of the pole $p^2 = m^2$ may differ from the mass parameter in the original Lagrangian. The key thing to take away from this result is that an isolated one-particle state corresponds to a pole in the complex p^0 plane.

2.2 LSZ for Fermions

We will apply this result to produce a LSZ formula for external fermion states. To motivate the formula, it will be instructive to compute a few expectation values in the free theory. After working in the free theory for a bit, we will come back to the full interacting theory.

Using our definition of states in the free theory, we find the inner product of one-particle states to be

$$ {}_a\langle k, s|p, r\rangle_a = (2\pi)^3 2p^0\, \delta^{rs}\, \delta^3(\vec{k} - \vec{p})\ , \tag{19} $$

where $|k, s\rangle_a = a^\dagger(k, s)|0\rangle$, and similarly for b-type particles. We also find the free-theory matrix elements

$$ \begin{aligned} \langle 0|\Psi_c(x)|p, s\rangle_a &= u_c(p, s)e^{-ip\cdot x} && (20)\\ \langle 0|\bar{\Psi}^d(x)|p, s\rangle_b &= \bar{v}^d(p, s)e^{-ip\cdot x} && (21)\\ {}_b\langle p, s|\Psi_c(x)|0\rangle &= v_c(p, s)e^{ip\cdot x} && (22)\\ {}_a\langle p, s|\bar{\Psi}^d(x)|0\rangle &= \bar{u}^d(p, s)e^{ip\cdot x}\ . && (23) \end{aligned} $$

The subscripts on the one-particle states refer to whether they are a-type (particles) or b-type (antiparticles). All the other possible combinations vanish. A handy way to think about these matrix elements is that we have contracted the field with the one-particle state.

An immediate implication of these matrix elements is that (in the free theory)

$$ {}_a\langle p_1, r| \left[\int d^4z\ \bar{\Psi}^c(z)\Psi_c(z)\right] |p_2, s\rangle_a = (2\pi)^4\delta^{(4)}(p_1 - p_2)\ \bar{u}^c(p_1, r)u_c(p_2, s) + \ldots\ , \tag{24} $$

where the omitted terms have $\Psi_c(z)$ and $\bar{\Psi}^c(z)$ contracted with each other. Let's compare this to the Fourier transform of the connected part of $\langle \Psi_{1a}\bar{\Psi}_2^b \int d^4z \bar{\Psi}_z^c\Psi_{zc}\rangle$:

$$ \begin{aligned} FT\langle \Psi_{1a}\bar{\Psi}_2^b \int d^4z\bar{\Psi}_z^c\Psi_{zc}\rangle_c &= \int d^4x_1\, e^{ip_1\cdot x_1} \int d^4x_2\, e^{-ip_2\cdot x_2} \langle \Psi_{1a}\bar{\Psi}_2^b \int d^4z\ \bar{\Psi}_z^c\Psi_{zc}\rangle && (25)\\ &= (2\pi)^4\delta^{(4)}(p_1 - p_2)\ \frac{i(\not{p}_1 + m)_a{}^c}{p_1^2 - m^2 + i\epsilon}\ \frac{i(\not{p}_2 + m)_c{}^b}{p_2^2 - m^2 + i\epsilon}\ . && (26) \end{aligned} $$

These are clearly different. However, suppose we multiply Eq. (26) by $(p_1^2 - m^2 + i\epsilon)\bar{u}^a(p_1, r)$ and $(p_2^2 - m^2 + i\epsilon)u_b(p_2, s)$, and sum over the a and b indices. Using

$$ (\not{p} + m)_a{}^c = \sum_{r'} u_a(p, r')\bar{u}^c(p, r'), \quad \text{and} \quad \bar{u}^a(p, r)u_a(p, r') = 2m\, \delta^{rr'}\ , \tag{27} $$

this gives

$$\frac{(-i)^2}{(2m)^2}(p_1^2-m^2+i\epsilon)\bar{u}^a(p_1,r)\,(p_2^2-m^2+i\epsilon)u_b(p_2,s)\;FT\langle\Psi_{1a}\bar{\Psi}_2^b\int d^4z\;\bar{\Psi}_z^c\Psi_{zc}\rangle$$
$$=\;(2\pi)^4\delta^{(4)}(p_1-p_2)\;\bar{u}^c(p_1,r)u_c(p_2,s)\;, \tag{28}$$

which is precisely the result of Eq. (24). In the free theory, at least, we now see how to connect the time-ordered products of field operators to particle states. If we had wanted antiparticle states instead of particle states, we would have instead multiplied by v^a and $\bar{v}^b$ as the projectors. For multiple initial- or final-state particles, we would have used more field operators and more projectors.

We can now give the LSZ reduction formula for fermions within the interacting theory. Instead of writing a big formula, it will be easier to state it as a series of operations. To compute the connected part of the scattering amplitude with m_a fermions and m_b antifermions in the initial state (with initial momenta $\{k_i\}$ and spins $\{s_i\}$) and n_a fermions and n_b antifermions in the final state (with final momenta $\{p_j\}$ and spins $\{r_j\}$):

1. Compute the connected part of the time-ordered vacuum expectation value of (m_a+n_b) $\bar{\Psi}$ fields and (m_b+n_a) Ψ fields using the master formula (or any other method you can think of).

2. Take the Fourier transform with respect to all the spatial coordinates. Use $\int\frac{d^4k_i}{(2\pi)^4}e^{-ik_i\cdot x_i}$ for the incoming momenta and $\int\frac{d^4p_j}{(2\pi)^4}e^{+ip_j\cdot x_j}$ for the outgoing momenta.

3. For each external state, multiply by the appropriate projector:

   - $-i(k^2-m^2)\left(\frac{1}{2m\sqrt{Z}}\right)u_{a_i}(k_i,s_i)$ for each incoming fermion.
   - $-i(p^2-m^2)\left(\frac{1}{2m\sqrt{Z}}\right)\bar{u}^{b_j}(p_j,r_j)$ for each outgoing fermion.
   - $-i(k^2-m^2)\left(\frac{1}{2m\sqrt{Z}}\right)\bar{v}^{b_i}(k,s_i)$ for each incoming antifermion.
   - $-i(p^2-m^2)\left(\frac{1}{2m\sqrt{Z}}\right)v_{a_j}(p_j,r_j)$ for each outgoing antifermion.

   Here, Z refers to the factor appearing in the one-particle portion of the fermion spectral formula, Eq. (18). Also, the Dirac indices on the spinors should match those of the corresponding fields and be summed over.

4. Take the limits $k_i^2\to m^2$ and $p_i^2\to m^2$.

The final result is the matrix element $\langle\{p_j,r_j\}|\{k_i,s_i\}\rangle_c$ in the interacting theory. Aside from the factors of $\sqrt{Z}$ (which are equal to unity at leading order in the perturbative expansion), these steps match what we did in the free theory. The new non-trivial physics content is that contributions from well-separated interacting particles in the initial and final states can still be identified with the poles in the n-point functions of the interacting theory.

To illustrate this procedure, let us apply it to the matrix element $\langle \phi(x_1)\Psi_a(x_2)\bar{\Psi}^b(x_3)\rangle$ we computed previously. Looking at Eq. (11), the signs we used in the Fourier transform correspond to p_1 and p_3 incoming and p_2 outgoing. Given our LSZ formula and the field content of the matrix element, this is the correct combination for an incoming scalar (p_1) and fermion (p_3), and an outgoing fermion (p_2). Applying the projectors, we get (to leading order)

$$\langle p_2|p_1p_3\rangle_c = (-iy)\frac{1}{Z_\Psi\sqrt{Z_\phi}}\,(2\pi)^4\delta^{(4)}(p_2-p_1-p_3)\;\bar{u}^c(p_2,r_2)u_c(p_3,s_3)\;. \tag{29}$$

Note that $Z_\Psi = Z_\phi = 1$ at this order, but we have written them here for posterity. By changing signs in the Fourier transforms and using different projectors, we could have also extracted from this matrix element the amplitudes for incoming and outgoing antifermions, or a fermion-antifermion pair in the initial or final state.

2.3 Feynman Rules for Scattering

While the LSZ formula is important conceptually and gives an algorithm to compute scattering matrix elements, it is much easier to compute them directly with a set of Feynman rules. These Feynman rules are very similar to those we formulated in momentum space for n-point functions, but with a different prescription for external legs.

The Feynman rules for our scalar-fermion theory are:

1. Draw all possible connected diagrams at whatever order in the coupling y you are interested in following the same rules as before. The only difference is in external fermion legs. For every incoming or outgoing fermion line, the direction of the line should follow the direction of the momentum flow. However, for every incoming or outgoing antifermion line, the direction of the line is *opposite* the direction of momentum flow. See Fig. 4 for an illustration. We do not distinguish between fermions and antifermions in internal lines, but the directions of the fermion lines should always match up.

2. Assign a value to every diagram:
   a) For all the internal pieces, everything goes through as before.
   b) For the external legs, follow the prescription given in Fig. 4.
   c) To get the Dirac index structure right, go backwards along each fermion line and contract the Dirac indices along the way. All these indices should be contracted in the final result.
   d) Multiply by $1\sqrt{Z_\Psi}$ for every external fermion and $1/\sqrt{Z_\phi}$ for every external scalar.
   e) Figure out the relative sign of the diagram by looking at the contraction structure. This turns out to be (-1) for every internal fermion loop and an additional relative sign for any two diagrams that differ by the exchange of fermion legs.

The result of all this is the scattering amplitude $-i\mathcal{M}$ with the fixed incoming momenta and spins $\{(k_i, s_i)\}$ and the outgoing momenta and spins $\{(p_j, r_j)\}$.

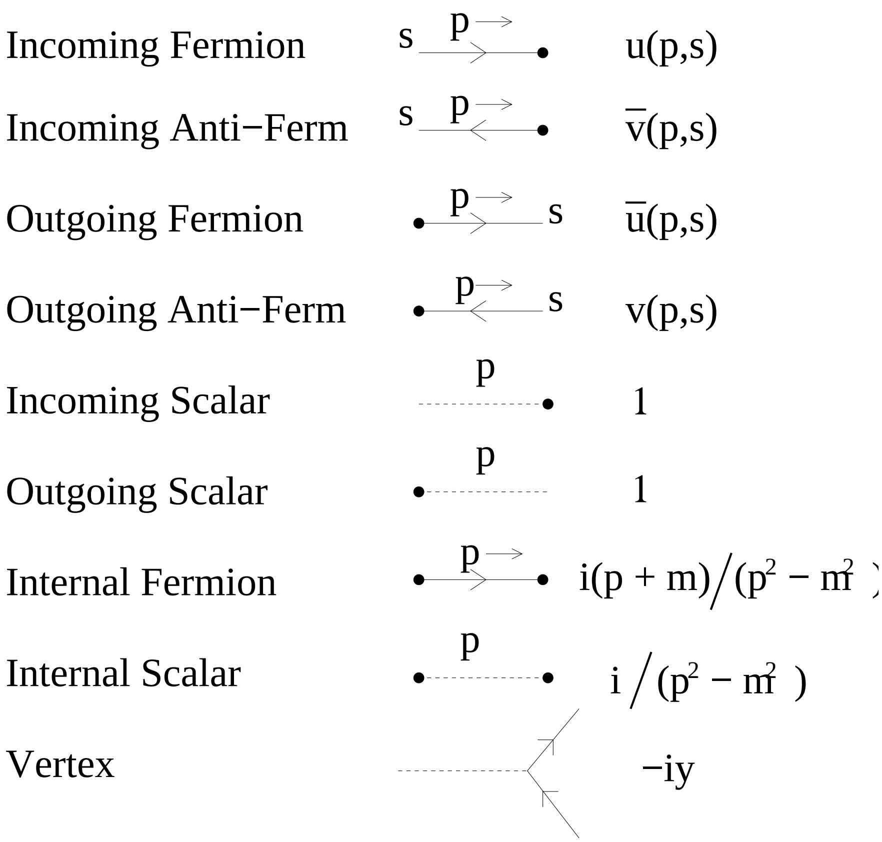

Figure 4: Feynman rules for the scalar-fermion theory.

References

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