PHYS 526 Notes #5: Poincaré and Particles

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So far in the course we have glossed over Lorentz invariance and we have only studied theories that describe particles with spin equal to zero. In these notes we will tackle Lorentz and translation invariance head on, and this will lead us to theories with particles of non-trivial spin.

Before addressing Lorentz invariance, we will first discuss the general features of symmetries and how to implement them in quantum mechanics. With this in hand, we will focus on the Poincaré group consisting of Lorentz transformations plus translations and which is the symmetry group of flat spacetime. In particular, we will study the implications of Poincaré invariance on the structure of quantum fields and particle states in the Hilbert space.

1 Symmetries, Groups, and Representations

Symmetry transformations obey the mathematical properties of a *group*, and it is worth spending a bit of time discussing what they are.[1] A group G is a set of objects together with a multiplication rule such that:

1. if $f,\, g \in G$ then $h = f\cdot g \in G$ (closure)
2. $f\cdot(g\cdot h) = (f\cdot g)\cdot h$ (associativity)
3. there exists an identity element $1 \in G$ such that $1\cdot f = f\cdot 1 = f$ for any $f \in G$ (identity)
4. for every $f \in G$ there exists an inverse element f^{-1} such that $f\cdot f^{-1} = f^{-1}\cdot f = 1$ (invertability)

Each group element corresponds to a different transformation of the same class.

A group can be defined via a multiplication table which specifies the value of $f\cdot g$ for every pair of elements $f,\, g \in G$. An *Abelian* group is one for which $f\cdot g = g\cdot f$ for every pair of $f,\, g \in G$. A familiar example of an Abelian group is the set of rotations in two dimensions. In contrast, the set of rotations in three dimensions is non-Abelian.

1.1 Representations of Groups

For the most part, we will be interested in symmetry transformations that act linearly on states in a Hilbert space,

$$|v_i\rangle \to |v_i'\rangle = U_{ij}|v_j\rangle. \tag{1}$$

[1] Much of this discussion is based on Refs. [1, 2], both of which provide a much more detailed account of the topics covered here.

As a result, we will usually work with matrix *representations* of groups. Groups themselves are abstract mathematical objects. A representation of a group is a set of $n \times n$ matrices $U(g)$, one for each group element, such that:

1. $U(f)U(g) = U(f \cdot g)$
2. $U(1) = \mathbb{I}$, the identity matrix.

Note that these conditions imply $U(f^{-1}) = U^{-1}(f)$. The value of n is called the dimension of the representation. For any group, there is always the *trivial* representation where $U(g) = \mathbb{I}$ for every $f \in G$. Note that a representation does not have to faithfully reproduce the full multiplication table. A representation is said to be *unitary* if all the representation matrices can be taken to be unitary ($U^\dagger = U^{-1}$).

***e.g.* 1.** Rotations in two dimensions
This group is formally called $SO(2)$ and can be defined as an abstract mathematical object. Any group element can be associated with a rotation angle θ. The most familiar representation is in terms of 2×2 matrices,

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

Of course, there is also the trivial representation.

Our focus will be primarily on continuous transformations. These correspond to what are called *Lie groups*, which are simply groups whose elements can be parametrized in terms of a set of continuous variables $\{\alpha^a\}$. We can (and will) always choose these coordinates (near the identity) such that the point $\alpha^a = 0$ corresponds to the identity element of the group. Thus, for any representation of the group, we have for infinitesimal transformations near the identity

$$U(\alpha^a) = \mathbb{I} - i\alpha^a t^a + \mathcal{O}(\alpha^2). \tag{3}$$

The matrices t^a are called *generators* of the representation. Finite transformations can be built up from infinitesimal ones according to

$$U(\alpha^a) = \lim_{p\to\infty} (1 - i\alpha^a t^a/p)^p = e^{-i\alpha^a t^a}. \tag{4}$$

This is nice because it implies that we only need to sort out a finite set of generators when discussing the representation of a Lie group rather than the infinite number of group elements. Let us also mention that a Lie group is said to be *compact* iff all the parameters α^a run over finite intervals.

A set of generator matrices $\{t^a\}$ can represent a Lie group provided they satisfy a *Lie algebra*. Besides being able to add and multiply them, they must also satisfy the following conditions:

1. $[t^a, t^b] = if^{abc}t^c$ for some constants f^{abc}
2. $[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0$ (Jacobi Identity)

The first condition is needed for the closure of the group (*i.e.* $\exp(-i\alpha^a t^a)\exp(-i\beta^a t^a) = \exp(-i\lambda^a t^a)$ for some λ^a) while the second is required for associativity. In fact, we can define a Lie group abstractly by specifying the *structure constants* f^{abc}. Most of the representations we'll work with are unitary, in which case the structure constants are all real and the generators t^a are Hermitian.

***e.g.* 2** $SU(2)$

This is the prototypical Lie group, and it should already be familiar from what you know about spin in quantum mechanics. As a group, it is defined to be the set of 2×2 unitary matrices with determinant equal to one. The corresponding Lie algebra has three basis elements which satisfy

$$[t^a, t^b] = i\epsilon^{abc}t^c \tag{5}$$

The basic *fundamental* representation of $SU(2)$ is in terms of Pauli matrices: $t^a = \sigma^a/2$. Since $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, it's clear that this is a valid representation of the algebra. You might also recall that any $SU(2)$ matrix can be written in the form $U = \exp(-i\alpha^a\sigma^a/2)$.

Even though $SU(2)$ is a group defined in terms of 2×2 matrices, it has other representations. You may recall from basic quantum mechanics that spin corresponds to a symmetry under $SU(2)$. The fundamental rep corresponds to $s = 1/2$, and $s = 0$ is just the trivial representation ($U(g) = \mathbb{I}$). On the other hand, we also know that there are spins with $s = 0,\ 1/2, 1,\ 3/2, \ldots$, and these correspond to represenations of dimension $(2s + 1)$.

1.2 More on Representations (optional)

The nice thing about working with linear generators t^a is that we can choose a nice basis for them. This is equivalent to choosing a nice set of coordinates for the Lie group. In particular, it is always possible to choose the generators t^a_r of any representation r such that

$$tr(t^a_r t^b_r) = T_2(r)\delta^{ab}. \tag{6}$$

The constant $T_2(r)$ is called the Dynkin index of the representation. With the exception of the Poincaré group (Lorentz and translations), we will always implictly work in bases satisfying Eq. (6), and we will concentrate on the case where the index is strictly positive. If so, the corresponding Lie group is said to be *compact* and is guaranteed to have finite-dimensional unitary representations. **This is not true of the Poincaré group, which is not compact.**

It turns out that there are only a finite set of classes of compact Lie groups. The *classical* groups are:

- $U(1)$ = phase transformations, $U = e^{i\alpha}$
- $SU(N)$ = set of $N \times N$ unitary matrices with $det(U) = 1$
- $SO(N)$ = set of orthogonal $N \times N$ matrices with $det(U) = 1$
- $Sp(2N)$ = set of $2N \times 2N$ matrices that preserve a slightly funny inner product.

In addition to these, there are the *exceptional* Lie groups: E_6, E_7, E_8, F_4, G_2. In studying the Standard Model, we will focus primarily on $U(1)$ and $SU(N)$ groups.

Some useful and fun facts about compact Lie algebras:

- Except for $U(1)$, we have $tr(t^a) = 0$ for all the classical and exceptional Lie groups.
- Number of generators = $d(G)$

$$d(G) = \begin{cases} N^2 - 1; & SU(N) \\ N(N-1)/2; & SO(N) \\ 2N(2N+1)/2; & Sp(2N) \end{cases} \tag{7}$$

- A representation (= rep) is *irreducible* if it cannot be decomposed into a set of smaller reps. This is true if and only if it is impossible to simultaneously block-diagonalize all the generators of the rep. Irreducible representation = irrep.
- If one of the generators commutes with all the others, it generates a $U(1)$ subgroup called an Abelian factor: $G = G' \times U(1)$.
- If the algebra cannot be split into sets of mutually commuting generators it is said to be *simple*. For example, $SU(5)$ is simple (as are all the classical and exceptional Lie groups given above) while $SU(3) \times SU(2) \times U(1)$ is not simple. In the latter case, all the $SU(3)$ generators commute with all the $SU(2)$ generators and so on.
- A group is *semi-simple* if it does not have any Abelian factors.
- With the basis choice yielding Eq. (6), one can show that the structure constants are completely anti-symmetric.
- The *fundamental* representation of $SU(N)$ is the set of $N \times N$ special unitary matrices acting on a complex vector space. This is often called the **N** representation. Similarly, the fundamental representation of $SO(N)$ is the set of $N \times N$ special orthogonal matrices acting on a real vector space.
- The *adjoint* (= A) representation can be defined in terms of the structure constants according to

  $$(t^a_A)_{bc} = -i f^{abc} \tag{8}$$

  Note that on the left side, a labels the adjoint generator while b and c label its matrix indices.

- Given any rep t^a_r, the conjugate matrices $-(t^a_r)^*$ give another representation, unsurprisingly called the conjugate representation. A rep is said to be real if it unitarily equivalent to its conjugate. The adjoint rep is always real.
- The Casimir operator of a rep is defined by $T^2_r = t^a_r t^a_r$ (with an implicit sum on a). One can show that T^2_r commutes with all the t^a_r. For an irrep (=irreducible representation) of a simple group, this implies that

$$T^2_r = C_2(r)\mathbb{I}, \tag{9}$$

for some positive constant $C_2(r)$.
- It is conventional to fix the normalization of the fundamental of $SU(N)$ such that $T_2(\mathbf{N}) = 1/2$. Once this is done, it fixes the normalization of all the other irreps. In particular, it implies that for $SU(N)$, $C_2(\mathbf{N}) = (N^2-1)/2N$, $T_2(A) = N = C_2(A)$.

2 Symmetries in Physical Systems

We have already discussed what it means for a transformation to be a symmetry of a classical system. Having discussed group representations, we can now apply this knowledge to choose convenient sets of variables for both classical and quantum systems.

2.1 Symmetries in Classical Mechanics

Recall that for a continuous classical system, the condition for the transformation

$$\phi_i \to \phi'_i = f_i(\phi) \tag{10}$$

to be a symmetry was

$$S[\phi] \to S[\phi'] := S'[\phi] = S[\phi]\ . \tag{11}$$

This usually implies that the Lagrangian should be unchanged by the transformation as well. Note as well that this is the *active picture*, and there is a similar relation for the passive picture.

In most cases of interest, we are interested in symmetry transformations that act linearly on the field variables. In this case, given a system described by n fields $\{\phi_1, \phi_2, \dots, \phi_n\}$, we have for each element g of the symmetry group

$$\phi_A(x) \to \phi'_A(x) = M_A{}^B(g)\,\phi_B(x)\ . \tag{12}$$

The matrices $M_A{}^B(g)$ form a representation of the symmetry group, in the sense described above, and there is a matrix for each element of the symmetry group. Note that these matrices $M(g)$ must be invertible, be they do not have to be unitary. For the Lagrangian to be invariant under the symmetry, it should be built up from terms that are also invariant. This strongly limits what can appear in the Lagrangian.

e.g. 3. A theory with $SU(2)$ symmetry.

Consider the theory of two of complex fields ϕ_1 and ϕ_2 combined to form a single two-component field Φ given by

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{13}$$

We will take the Lagrangian for the theory to be

$$\mathscr{L} = (\partial^\mu \Phi^\dagger)(\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{\lambda}{4}(\Phi^\dagger \Phi)^2 \tag{14}$$

This theory is invariant under $SU(2)$ transformations

$$\Phi(x) \to \Phi'(x) = e^{-i\alpha^a t^a}\Phi , \tag{15}$$

where $t^a = \sigma^a/2$ are the Pauli matrices. In this case, $M(\alpha^a) = e^{i\alpha^a\sigma^a/2}$. This is only one of many different theories with a symmetry under $SU(2)$ transformations. For instance, we could have also constructed a theory using the triplet representation of $SU(2)$ where the t^a matrices are proportional to the matrices that arise when you construct the spin operators S_x, S_y, and S_z on states of spin $s = 1$. (Note that the $SU(2)$ in this example has nothing to with spin!)

2.2 Symmetries in Quantum Mechanics

In quantum mechanics we have states in a Hilbert space and operators that act upon them. Observables correspond to the eigenvalues of Hermitian operators, and the squared norms of states are interpreted as probabilities. In general, a transformation upon a quantum mechanical system can be implemented by an operator acting on all states:

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle , \tag{16}$$

for some operator U. By construction, this operator is linear.

The transformations that tend to be the most interesting are symmetries. We already discussed what it means for a transformation to be a symmetry of a classical system. In quantum mechanics, we demand that a symmetry (not) do two things. The first is that it should not alter the inner products of states (which are interpreted as probabilities),

$$\langle\psi_a|\psi_b\rangle = \langle\psi'_a|\psi'_b\rangle \tag{17}$$

$$= \langle\psi_a|U^\dagger U|\psi_b\rangle , \tag{18}$$

which implies that we need $U^\dagger U = \mathbb{I}$. Thus, the transformation should be implemented by a unitary operator.[2] We would also like the transformation to be consistent with time evolution. In the Schrödinger picture, this means specifically that

$$|\psi'(t)\rangle = e^{-iH(t-t_0)}|\psi'(t_0)\rangle \tag{19}$$

$$Ue^{-iH(t-t_0)}|\psi(t_0)\rangle = e^{-iH(t-t_0)}U|\psi(t_0)\rangle , \tag{20}$$

[2] Actually, this is a bit too strict since equality of the inner products up to a phase would have been fine as well. However, we will concentrate on unitary transformations in this course.

which implies that[3]

$$[H, U] = 0 \ , \tag{21}$$

Together, Eqs (18,21) are the conditions that a set of transformations must satisfy to be a symmetry in quantum mechanics.

In general, the set of all symmetry transformations on the system forms a group. For each element g in the group, there is an operator $U(g)$ that acts linearly on the vector space of states. Therefore they must be a linear representation of the group. It is often useful to choose a basis of states that transform as independent blocks under the symmetry. For example, with spin we decompose the Hilbert space into states with different spins.

If we apply this definition to a continuous symmetry, we immediately get a quantum version of Noether's theorem. A continuous symmetry corresponds to a Lie group, and thus we can write any group element in terms of the generators of the Lie algebra as

$$U(\alpha^a) = e^{-i\alpha^a t^a} \ , \tag{22}$$

where α^a are the continuous parameters that label the different elements of the group and now t^a is a quantum operator that represents the Lie algebra of the group ($[t^a, t^b] = i f^{abc} t^c$). By assumption, $U(\alpha^a)$ commutes with the Hamiltonian for any value of α^a. This implies that

$$[H, t^a] = 0 \ . \tag{23}$$

In the Heisenberg picture, this implies that the generators are conserved in that they do not evolve in time.

Just like we had the Schrödinger and Heisenberg pictures of time evolution in quantum mechanics, or the active and passive pictures of symmetries in classical mechanics, we can also think of symmetries as acting on operators rather than on states. It turns out that this is usually the more convenient thing to do in quantum field theories. In this picture, a symmetry transformation corresponds to keeping all the states the same but transforming the operators according to

$$\mathcal{O} \to \mathcal{O}' = U^\dagger(g) \mathcal{O} U(g) \ , \tag{24}$$

where $U(g)$ is unitary with $[H, U(g)] = 0$ as before. For continuous symmetries parametrized by α^a, we can consider and infinitessimal transformation and define

$$\mathcal{O}' - \mathcal{O} = \alpha^a \, (\Delta\mathcal{O})^a \tag{25}$$

Expanding $U(\alpha^a)$ and matching up α^a factors, we find

$$(\Delta\mathcal{O})^a = i[t^a, \mathcal{O}] \ . \tag{26}$$

This implies that the action of the transformation on the operator is encoded in its commutator with the generators (analagous to the commutator with H giving the time evolution).

[3]This assumes that U has no explicit time dependence. If it does, the condition is $-i\partial_t U + [H, U] = 0$.

In quantum field theories, we will usually be interested in symmetries that act linearly on the fields. By this, we mean that for a system of n fields $\{\phi_1, \ldots, \phi_n\}$ we have

$$\phi_A(x) \to \phi'_A(x) = U^\dagger(g)\phi_A(x)U(g) = M_A^{\ B}(g)\phi_B(x) \ , \tag{27}$$

where the matrices M form a representation of the symmetry group.[4] Note that we have three levels of mathematical structure here. The first is the abstract structure of the symmetry group itself, which need not refer to any matrices at all. The second level is the representation of the group on the states of the Hilbert space by the unitary operators U. And the third level is the representation of the group by the M matrices acting on the space fields. These matrices do not have to be unitary.

It is worth checking that this picture is consistent. We have trivially that $U(1) = \mathbb{I}$ and $M_A^{\ B}(1) = \delta_A^{\ B}$, and associativity and invertability follow by assumption. The last thing to check is closure. Suppose we transform by the group element $f \cdot g$. This gives on the left-hand side of Eq. (27)

$$\begin{aligned} U^\dagger(f\cdot g)\phi_A(x)U(f\cdot g) &= U^\dagger(g)U^\dagger(f)\phi_A(x)U(f)U(g) && (28)\\ &= U^\dagger(g)\left[M_A^{\ B}(f)\phi_B(x)\right]U(g) && (29)\\ &= M_A^{\ B}(f)M_B^{\ C}(g)\phi_C(x) \ . && (30)\end{aligned}$$

On the right-hand side, we find

$$M_A^{\ C}(f\cdot g)\phi_C(x) = M_A^{\ B}(f)M_B^{\ C}(g)\phi_C(x) \ . \tag{31}$$

Thus, everthing works out as it should.

3 The Poincaré Group: Lorentz plus Translations

The symmetries we are most interested in for relativistic theories are those of spacetime, namely translations and Lorentz transformations (boosts and rotations). Together, this group of symmetries is called the Poincaré group. When we get to finding representations of the Poincaré group on fields, we will see that they correspond to particles of different spins.

3.1 The Translation Group

The Poincaré group consists of translations plus Lorentz transformations. We have already discussed translations,

$$x^\mu \to x^\mu + a^\mu \ . \tag{32}$$

Translations form a group that can be parametrized by the four components of the translation vector a^μ. We can represent the group on the space of functions of x by the linear operator

$$U(a) = e^{ia^\nu(i\partial_\nu)} \ , \tag{33}$$

[4] If the transformation shifts $x \to x'$ as well, this result is modified to $\phi'(x') = U^\dagger\phi(x')U = M\phi(x)$.

that produces

$$U(a)f(x) = \left[1 - a^\nu\partial_\nu + \frac{1}{2!}(-a^\nu\partial_\nu)^2 + \ldots\right] f(x) = f(x-a) \; . \tag{34}$$

Specializing to infinitesimal translations, we find the generators of this representation to be

$$P_\mu = i\partial_\mu \; . \tag{35}$$

These satisfy the Lie algebra

$$[P^\mu, P^\nu] = 0 \; . \tag{36}$$

The group of translations is formally defined as the abstract Lie group spanned by the parameters $a^\mu \in (-\infty, \infty)$ with Eq. (36) as the Lie algebra. Our labelling of the generators by P^μ reflects our result that the conserved currents corresponding to invariance under translations are the energy and momentum operators.

3.2 The Lorentz Group

The second component of the Poincaré group are Lorentz transformations. These are defined as the set of linear transformations of spacetime that leave the Lorentz interval unchanged. Specifically, a Lorentz transformation is any real linear transformation Λ such that

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \; , \tag{37}$$

with

$$\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \; . \tag{38}$$

This implies that the transformation matrices must satisfy

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \; , \tag{39}$$

which is equivalent to the condition

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu := \eta_{\nu\beta}\, \eta^{\mu\alpha}\, \Lambda^\beta{}_\alpha \; . \tag{40}$$

Besides just x^μ, we call any four-component object v^μ transforming this way under Lorentz a *four vector*:

$$v^\mu \to v'^\mu = \Lambda^\mu{}_\nu v^\nu \; . \tag{41}$$

The result of Eq. (40) implies that the dot product of any pair of four vectors v^μ and w^μ is unchanged by Lorentz transformations,

$$v \cdot w := \eta_{\mu\nu} v^\mu w_\nu = v' \cdot w' \; . \tag{42}$$

More generally, we define an $(n, 0)$-index Lorentz tensor to be an object $T^{\mu_1\mu_2\cdots\mu_n}$ that transforms as

$$T^{\mu_1\ldots\mu_n} \to T'^{\mu_1\ldots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\ldots\Lambda^{\mu_n}{}_{\nu_n}T^{\nu_1\ldots\nu_n} \ . \tag{43}$$

From this point of view, a vector is just a $(1, 0)$ tensor. We also define tensors with lowered indices by starting with a $(n, 0)$ tensor and lowering some of the indices with $\eta_{\mu\nu}$. We call a tensor with n upper indices and m lower indices a (n, m) tensor. It is straightforward to show that any product of tensors with all the indices contracted is Lorentz-invariant. More generally, any quantity defined on spacetime can be decomposed into Lorentz tensors as far as its Lorentz transformation properties are concerned.

Let us turn next to the Lie group structure of Lorentz transformations. Since this group is defined in terms of a set of linear transformations on 4-vectors, we already know one representation of the group. By matching these transformations to the general structure of Lie groups, we can figure out how to build other represenations. The identity element of the group is clearly $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$. Expanding around the identity, we have

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \ldots \tag{44}$$

If we plug this into the requirement of Eq. (38), we find that the the only condition on $\omega^\mu{}_\nu$ is that it be antisymmetric:

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \ . \tag{45}$$

With this constraint, $\omega^\mu{}_\nu$ has six independent elements that we can identify with the three generators of spatial rotations and the three generators of Lorentz boosts.

To put this another way, we have just seen that the Lorentz group can be parametrized in terms of six real numbers, which we can write as a antisymmetric $(0, 2)$ tensor $\omega_{\mu\nu}$. A general unitary reprentation of the Lorentz group must therefore take the form

$$U(\omega) = e^{-i\omega_{\mu\nu}J^{\mu\nu}/2} = \mathbb{I} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + \ldots \ , \tag{46}$$

where the $J^{\mu\nu}$ are a set of six antisymmetric Hermitian operators that generate the group representation.[5] As an operator, consistency of the representation requires that $J^{\mu\nu}$ transforms as a $(2, 0)$ tensor,

$$U^\dagger(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma} \ . \tag{47}$$

Expanding this result out when $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ is also infinitesimal gives the commutation relations of the $J^{\mu\nu}$ generators:

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= -\frac{i}{2}\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\mu\sigma}J^{\nu\rho}\right) & (48)\\ &= -i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\sigma}J^{\mu\rho}\right) \ . & (49)\end{aligned}$$

[5]The factor of two here is just a matter of tradition.

These relations define the abstract Lie algebra of the Lorentz group.

The six independent $J^{\mu\nu}$ generators can be rewritten in terms of the more familiar generators of rotations and boosts. Let us define

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}\ , \qquad K^i = J^{0i}\ . \tag{50}$$

The commutation relations of Eq. (49) then imply

$$[J^i, J^j] = i\epsilon^{ijk}J^k\ , \tag{51}$$
$$[K^i, K^j] = -i\epsilon^{ijk}J^k\ , \tag{52}$$
$$[J^i, K^j] = i\epsilon^{ijk}K^k\ . \tag{53}$$

The first should be familiar, while the second and third are the generalizations to boosts.

With the Lie algebra of Lorentz in hand, we can try to find representations of it. In fact, we already have one, namely the Lorentz group as a set of linear operators acting on the space of four vectors. It is not hard to check that the generators of this representation are

$$(J_4^{\mu\nu})_{\alpha\beta} = i(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha)\ . \tag{54}$$

Here, μ and ν label which generator we want, while α and β label the elements of the representation matrix. Explicitly, we have

$$\Lambda^\alpha{}_\beta = \left(\exp\left[-\frac{i}{2}\omega_{\mu\nu}(J_4^{\mu\nu})\right]\right)^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(J_4^{\mu\nu})^\alpha{}_\beta + \ldots \tag{55}$$
$$= \delta^\alpha{}_\beta + \omega^\alpha{}_\beta + \ldots\ , \tag{56}$$

just like Eq. (44). Note that for any Λ, there exists a unique ω. A second simple representation is the Lorentz group acting on the space of functions of x. The generators are

$$J_x^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\ . \tag{57}$$

These generalize the familiar angular momentum operators $\vec{J} = \vec{x}\times(-i\vec{\nabla})$ in position-space quantum mechanics.

There's a low-down dirty trick to find the general representations of the Lorentz group. Let us define new operators A^i and B^i by

$$A^i = \frac{1}{2}(J^i + iK^i)\ , \qquad B^i = \frac{1}{2}(J^i - iK^i)\ . \tag{58}$$

These operators have the commutation relations

$$[A^i, A^j] = i\epsilon^{ijk}A^k \tag{59}$$
$$[B^i, B^j] = i\epsilon^{ijk}B^k \tag{60}$$
$$[A^i, B^j] = 0\ . \tag{61}$$

Note as well that $(A^i)^\dagger = B^i$ and so on. These factors each have the commutation relations of two independent $SU(2)$ Lie algebras. Thus, to find representations of Lorentz, all we need to do is specify the representations of each of the $SU(2)$ factors. The irreducible representations are therefore (j_A, j_B), with j_A and j_B half-integer, and have dimension $(2j_A+1)\times(2j_B+1)$.

The lowest representation is just the trivial $(0,0)$. It corresponds to the scalars we have been studying so far. The next two up are $(1/2,0)$ and $(0,1/2)$, both of dimension two. We will see that these correspond to left- and right-handed *Weyl fermions*. Under the rotation subgroup of Lorentz, they both unsurprisingly have spin $j = 1/2$. The next representation is the four-dimensional $(1/2,1/2)$. It corresponds to a Lorentz vector, and it decomposes into spins $j = 0,\,1$ under the rotation subgroup. Going to higher values of j_A and j_B gives even higher spins.

In the discussion so far, we have implicitly assumed that the elements of the Lorentz group can be continuously deformed to the identity. This isn't necessarily true. Two matrices that satisfy the condition of Eq. (39) that cannot be deformed smoothly to unity are

$$\mathcal{P} = diag(+1,-1,-1,-1)\ , \tag{62}$$

and

$$\mathcal{T} = diag(-1,+1,+1,+1)\ . \tag{63}$$

The matrix $\mathcal{P}$ is called a parity transformation and T is called time reversal. It turns out that any Lorentz transformation can be written as the product of a transformation connected to the identity times either $\mathcal{P}$, $\mathcal{T}$, or $\mathcal{PT}$. In this sense, the Lorentz group has four independent sectors. We will be interested mainly in the sector connected to the identity, called the *proper orthochronous* subgroup of Lorentz, which is often what is meant by "Lorentz group", and from here on we will follow this convention. In Nature, it turns out that "Lorentz" is a good symmetry but parity and time-reversal are not.

Applying $\mathcal{P}$ to the generators, we find that $J^i \to J^i$ and $K^i \to -K^i$. Correspondingly, we find $A^i \leftrightarrow B^i$. Thus, the effect of parity is to map $(j_A, j_B) \leftrightarrow (j_B, j_A)$. The Weyl spinor irreps of Lorentz are therefore not representations of the Lorentz group extended by parity. On the other hand, the reducible representation $(1/2,0)\oplus(0,1/2)$ does work. We will use this rep soon to describe electrons in QED.

3.3 The Poincaré Group

The full Poincaré group consists of translations and Lorentz, and the general form of such a transformation is

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu\ . \tag{64}$$

Thus, we will label general Poincaré group elements by

$$\{\Lambda,\, a\}\ . \tag{65}$$

We would like to find representations of this larger group. To do so, we need to figure out the Lie algebra.

Since we already have the Lie algebras for the translation and Lorentz subgroups, the only other things we need are the commutation relations between P^μ and $J^{\rho\sigma}$, provided they don't induce any new operators. These can be obtained by applying a Lorentz transformation to the P^μ operator, which must transform like a 4-vector,

$$U^\dagger(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \ . \tag{66}$$

Specializing to $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \ldots$ and $U(\Lambda) = \mathbb{I} - i\omega_{\rho\sigma}J^{\rho\sigma}/2 + \ldots$, we find

$$[P^\mu, J^{\rho\sigma}] = i\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right) \ . \tag{67}$$

Collecting this and our previous results, the full set of commutation relations for the Poincaré group is therefore

$$[P^\mu, P^\nu] = 0 \tag{68}$$
$$[P^\mu, J^{\rho\sigma}] = i\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right) \tag{69}$$
$$[J^{\mu\nu}, J^{\rho\sigma}] = -i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\sigma}J^{\mu\rho}\right) \ , \tag{70}$$

Together, these relations completely fix the Lie algebra structure of the Poincaré group.

Looking at Eq. (69), it would appear that some of the $J^{\mu\nu}$ currents are not conserved, in that $[P^0, J^{\mu\nu}] \neq 0$ (recall that $P^0 = H$). Working out the details, the non-zero components correspond to the boosts,

$$[H, J^{0i}] = iP^i \ . \tag{71}$$

This contradicts Eq. (21), one of our conditions for an operator to be the generator of a symmetry. It turns out that Eq. (21) does not quite apply to the case where the operator has an explicit dependence on t (*i.e.* t appears in the operator on its own, and not as the argument of a field). The correct generalization is

$$0 = i[H, U] + \frac{\partial U}{\partial t} = \frac{dU}{dt} \ . \tag{72}$$

Working out the Noether current for $J^{\mu\nu}$ in a general Lorentz-invariant field theory, one finds that the J^{0i} components depend on t explicitly and satisfy the general condition of Eq. (72) [4].

3.4 Representations of Poincaré on Fields

To construct a Poincaré-invariant quantum field theory, it is convenient to use field variables that have well-defined transformation properties, corresponding to fields that transform under definite representations of the Poincaré group. Note that unlike the representations of groups on quantum mechanical states, the representations of Poincaré on fields do not have to be unitary.

Given a Poincaré transformation

$$x \to x' = \Lambda x + a \ , \tag{73}$$

the corresponding transformation on a field operator will be

$$\phi_A(x) \to \phi'_A(x) = U^\dagger(\Lambda, a)\phi_A(x)U^\dagger(\Lambda, a) \ , \tag{74}$$

with the transformation operator

$$U(\Lambda, a) = \mathbb{I} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + ia_\mu P^\mu + \ldots \tag{75}$$

Our challenge is to figure out what $\phi'_A(x)$ can be.

Specializing to the translation subgroup of Poincaré, we already know the result. Recall that under $x \to x' = x + a$, we had

$$\phi_A(x) \to \phi'_A(x) = \phi_A(x-a) = e^{-iP\cdot a}\phi_A(x)e^{iP\cdot a} \ . \tag{76}$$

This has precisely the form of Eq. (74) with

$$U(1, a) = e^{ia_\mu P^\mu} \ , \tag{77}$$

where now the Hermitian P^μ operators act on the Hilbert space.

Let us turn next to the Lorentz subgroup. The most simple representation of Lorentz is the trivial one, with $\phi(x) \to \phi'(x') = \phi(x)$. This implies

$$\phi(x) \ \ \to \ \ \phi'(x) = U^\dagger(\Lambda, 0)\phi(x')U(\Lambda, 0) = \phi(\Lambda^{-1}x) \ . \tag{78}$$

Put another way, the shifted field is the same function of the rotated coordinates as the original field on the original coordinates. For example, if $\phi(x)$ is zero everywhere except at $x = x_0$, $\phi'(x)$ will be zero everywhere except at $x'_0 = \Lambda x_0$ with $\phi'(\Lambda x_0) = \phi(x_0)$. This has the form of Eq. (27), but with an additional change in the x coordinate. We found previously that this trivial representation corresponds to particles of spin $s = 0$.

A simple generalization of this result is the Lorentz vector field A^μ. Here, the four component fields rotate into each other under a Lorentz transformation. More precisely,

$$A^\mu(x) \to A'^\mu(x) = U^\dagger(\Lambda, 0)A^\mu(x)U(\Lambda, 0) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{79}$$

We see that the field components are rotated in addition to the shift in the coordinate dependence. They clearly transform in the vector representation of Lorentz. This also has the form of Eq. (27) with the representation matrix $M_A{}^B \sim \Lambda^\mu{}_\nu$.

It is instructive to look at the infinitessimal form of this transformation, with $\Lambda = 1 + \omega$. Expanding the left side of Eq. (79) we find

$$U^\dagger(1+\omega, 0)A^\mu(x)U(1+\omega, 0) = A^\mu(x) + \frac{i}{2}\omega_{\alpha\beta}[J^{\alpha\beta}, A^\mu(x)] \ . \tag{80}$$

On the right side of Eq. (79), we find

$$\Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x) = A^{\mu}(x) - \frac{i}{2}\omega_{\alpha\beta}\left[i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha})A^{\mu}(x) + i(\eta^{\mu\alpha}\delta^{\beta}{}_{\nu} - \eta^{\mu\beta}\delta^{\alpha}{}_{\nu})A^{\nu}(x)\right] \ . \tag{81}$$

Comparing both sides, we see that

$$i[J^{\alpha\beta}, A^{\mu}(x)] = -iJ_{x}^{\alpha\beta}A^{\mu}(x) - i(J_{4}^{\alpha\beta})^{\mu}{}_{\nu}A^{\nu}(x) \ . \tag{82}$$

Comparing to the general result of Eq. (26), the left side just says that this is the variation of the $A^{\mu}(x)$ operator under infinitesimal Lorentz transformations. On the right side, there are two contributions. The first term is the variation due to the shift in the spacetime coordinate x. The second term is due to the vector transformation of the field components. Specializing to the ij components, this is just the expression for the infinitesimal change in the vector field under spatial rotations. In this case, the first term corresponds to the orbital angular momentum carried by the field while the second is due to the intrinsic "spin" of the field. Note that there is no spin contribution for $\mu = 0$.

Given this result for the vector field, the generalization to an arbitrary representation of the Lorentz subgroup acting on the fields $\{\phi_C\}$ should not be too suprising. It is

$$i[J^{\alpha\beta}, \phi_C(x)] = -iJ_{x}^{\alpha\beta}\phi_C(x) - i(J_{AB}^{\alpha\beta})_{C}{}^{D}\phi_D(x) \ , \tag{83}$$

where $J_{AB}^{\alpha\beta}$ is the generator of the (j_A, j_B) representation of the Lorentz Lie algebra discussed in Section 3.2. This relation gives the infinitessimal variation in the field $\phi_C(x)$ under Lorentz. Composing them, we can build up the general form of a finite Lorentz transformation:

$$U^{\dagger}(\Lambda,0)\phi_C(x)U(\Lambda,0) = \left[\exp(-\frac{i}{2}\omega_{\alpha\beta}J_{AB}^{\alpha\beta})\right]_{C}^{\ D}\phi_D(\Lambda^{-1}x) \ . \tag{84}$$

We can now combine translations and Lorentz transformations and give the general form a finite Poincaré transformation on a field. It is just

$$U^{\dagger}(\Lambda,a)\phi_C(x)U(\Lambda,a) = \left[\exp(-\frac{i}{2}\omega_{\alpha\beta}J_{AB}^{\alpha\beta})\right]_{C}^{\ D}\phi_D(\Lambda^{-1}x - a) \ . \tag{85}$$

The transformation matrix on the right-hand side describes the representation of the field under Lorentz, while the change in the coordinate of the field contains the rest of Lorentz together with all of Poincaré.

3.5 Representations of Poincaré on States

Finding the representations of the Poincaré group on one-particle quantum states is slightly more challenging. The difficulty lies in the fact that we are only allowed to use unitary representations. This has a very important consequence: physical particle states with mass $m > 0$ can be characterized completely by their mass and their representation under the

$SU(2)$ spin group. In contrast, massless particles states can be characterized by their representation under a slightly different group, called helicity. For more details, see Refs. [5, 6, 7].

The translation part of the Poincaré group is trivial to include if we work with simultaneous eigenstates of P^μ:

$$P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle \ , \tag{86}$$

where the σ index labels any other properties the state could have under Lorentz. Applying a Lorentz transformation to such a state, we must have

$$U(\Lambda)|p,\sigma\rangle = [D_p(\Lambda)]_{\sigma\sigma'}|\Lambda p,\sigma'\rangle \ , \tag{87}$$

where $[D_p]_{\sigma\sigma'}$ is a matrix describing how states transform into each other under Lorentz. Our remaining task is to figure out what these matrices can be.

Since we are using momentum eigenstates, these states are also eigenstates of the $P^2 = P_\mu P^\mu$ operator. This operator is invariant under both translations and Lorentz, which can be verified by checking that it commutes with all the Poincaré generators P^μ and $J^{\mu\nu}$. Thus, for a set of states transforming under a given representation of Poincaré, they must all have the same eigenvalue of P^2. For this reason, P^2 is called a *Casimir invariant* of the group and its eigenvalues are usually labelled by $P^2 = M^2$. If we also use the fact that (proper orthochronous) Poincaré transformations do not change the sign of p^0, we can subdivide the possible representations into nine subclasses according to whether M^2 and p^0 are positive, negative, or zero. The cases of physical relevance are $M^2 = 0 = p^0$, $M^2 > 0$ and $p^0 > 0$, and $M^2 = 0$ and $p^0 > 0$. The first of these corresponds to the vacuum $|\Omega\rangle$ which transforms under the trivial representation. The second and third correspond to massive and massless particles.

For a massive particle with mass M, let us define a reference momentum

$$k^\mu = (M,\vec{0}) \ , \tag{88}$$

corresponding to the rest frame of the particle. We can get to any other momentum p (with $p^2 = M^2$) in a unique way by applying the Lorentz transformation L_p. On states, this implies that

$$|p,\sigma\rangle = U(L_p)|k,\sigma\rangle \ . \tag{89}$$

Note that the same index σ appears on both sides of the equation. This corresponds to a specific choice for how to relate the index of the reference state to the index of the more general state.

For arbitrary transformations, the transformed state will have a modified index structure as in Eq. (87). Performing another Lorentz transformation Λ, we get (leaving out the σ stuff for now)

$$U(\Lambda)|p\rangle = U(\Lambda)U(L_p)|k\rangle \tag{90}$$

$$= U(L_{\Lambda p})U^\dagger(L_{\Lambda p})U(\Lambda)U(L_p)|k\rangle \tag{91}$$

$$= U(L_{\Lambda p})\,U(L_{\Lambda p}^{-1}\Lambda L_p)|k\rangle \ . \tag{92}$$

This simple rearrangement has an important physical implication. The argument of the second U operator in the last line is a Lorentz transformation that takes $k \to p \to \Lambda p \to k$. It is therefore an element of the *little group* of Lorentz, the subgroup of Lorentz that maps k^μ to itself. To figure out the effect of the transformation on the σ indices, it is therefore sufficient to find the representations of the little group alone.

Given the form of the reference momentum in Eq. (88), the little group for the case of a massive particle is just the spatial rotation subgroup of Lorentz generated by the J^a, $a = 1,2,3$. This subgroup is just $SU(2)$, and we can therefore identify the σ label with a spin index! The unitary one-particle representations of a massive particle are therefore labelled by its 4-momentum and spin. This might not seem too surprising, but here we have found it to be a consequence of the underlying Poincaré symmetry.

We also know how to build represenatations of $SU(2)$, and this allows us to deduce the form of the $[D_p(\Lambda)]$ matrices. For any element $W^\mu_{\ \nu}$ of the little group ($W^\mu_{\ \nu}k^\nu = k^\mu$), we have

$$U(W)|k,\sigma\rangle := [d(W)]_{\sigma\sigma'}|k,\sigma'\rangle \ , \tag{93}$$

where the little group matrix is

$$d(W) = e^{-i\theta^a J^a_j} \ , \tag{94}$$

for some spin-j representation J^a_j of $SU(2)$. The generators therefore have dimension $(2j+1)\times(2j+1)$ and the σ index can be identified with m_j values (for states $|j,m_j\rangle$).

Applying this to a general Lorentz transformation on a general state,

$$\begin{aligned}
U(\Lambda)|p,\sigma\rangle &= U(L_{\Lambda p})\,U(W(\Lambda,p))|k,\sigma\rangle && (95)\\
[D_p(\Lambda)]_{\sigma\sigma'}\,|\Lambda p,\sigma'\rangle &= U(L_{\Lambda p})[d(W(\Lambda,p)]_{\sigma\sigma'}|k,\sigma'\rangle && (96)\\
&= [d(W(\Lambda,p)]_{\sigma\sigma'}|\Lambda p,\sigma'\rangle \ . && (97)
\end{aligned}$$

Comparing both sides, we see that $D_p(\Lambda) = d(W(\Lambda,p))$, where $W(\Lambda,p)$ is the combined little group Lorentz transformation that sends $k \to p \to \Lambda p \to k$.

The total spin of a representation can be related to a second Casimir operator. Let us define the *Pauli-Lubanski pseudovector* by

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma \ , \tag{98}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon^{0123} = +1$. Since $W_\mu P^\mu = 0$, in the particle rest frame we have $W^\mu = (0,\vec{W})$ with

$$W^i = M\,J^i \ . \tag{99}$$

The quantity $W^2 = W_\mu W^\mu$ is therefore Lorentz invariant and equal to

$$W^2 = -M^2\vec{J}\cdot\vec{J} \to -M^2 s(s+1) \ . \tag{100}$$

With a bit of work, you can show that W^2 also commutes with all the Poincaré generators. Thus, it is a Casimir invariant of the group and is equal to a fixed number in any representation. In all, we can completely charaterize any massive representation of Poincaré by the mass M and total spin s.

These arguments go through in the same way in the massless case with one important difference; a massless particle has no rest frame. Instead, a reasonable choice for the reference momentum is

$$k^\mu = (k, 0, 0, k) \ . \tag{101}$$

The little group is no longer the rotation group. Instead, it is generated by J^3, $L_1 = K^1 - J^2$, and $L_2 = K^2 + J^1$, which have the Lie algebra of a different group called $ISO(2)$. It turns out that the only finite-dimensional representation of this group has $W^2 = 0$ along with $P^2 = 0$. This implies that

$$W^\mu|\vec{k}\rangle = hP^\mu|\vec{k}\rangle \tag{102}$$

on any such massless state. The proportionality constant is called the *helicity* of the particle, and is equal to

$$h = \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|} = W^0/|\vec{p}| \ . \tag{103}$$

This single number characterizes the finite representation of the little group, and must either be integer or half-interger. It turns out that the consistency of a theory requires that it contain states with both positive and negative helicity. These two helicity eigenstates correspond to "spin" parallel ($h > 0$) and antiparallel ($h < 0$) to the direction of motion. This result has important implications for the photon. First, it agrees with the fact that the photon only has two independent polarizations. Second, we will see that the one-particle state of the photon will be created by a vector field A^μ with four components, only two of which can correspond to physical excitations. Getting rid of the extra degrees of freedom will turn out to be highly non-trivial and will lead us to gauge invariance.

4 Executive Summary

Here's the short version of all this:

- Symmetry transformations have the mathematical structure of a group.
- A representation of a group is a set of linear operators $M(g)$ (which can be written as matrices once we specify a basis) that obey the group multiplication rules: $M(1) = \mathbb{I}$ and $M(f \cdot g) = M(f)M(g)$. Even though the representation is properly the set of linear operators, sometimes the vector space upon which they act is also called "the representation" of the group.
- A Lie group is one that can be parametrized by continuous coordinates $\{\alpha^a\}$. For group elements connected to the identity, we have $U(\alpha^a) = \exp(-i\alpha^a t^a)$. To represent such elements, we only need to find a representation of the Lie algebra for t^a.
- In quantum mechanics, symmetries are implemented by unitary operators $U(g)$ acting on states with $[H, U] = 0$. Equivalently, we can transform the operators instead of the states according to $\mathcal{O} = U^\dagger(g)\mathcal{O}U(g)$. For Lie groups, this implies that $\partial_t t^a = 0$ is conserved and $(\Delta\mathcal{O})^a = i[t^a, \mathcal{O}]$.
- Poincaré = translations plus Lorentz. The corresponding Lie algebra is

$$[P^\mu, P^\nu] = 0 \tag{104}$$

$$[P^\mu, J^{\rho\sigma}] = i\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right) \tag{105}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\sigma}J^{\mu\rho}\right) . \tag{106}$$

  Translations are easy to represent using functions of spacetime. For Lorentz, we can rewrite the generators as a pair of $SU(2)$ factors, and the representations are labelled by (j_A, j_B) with $j_{A,B} = 0,\ 1/2,\ 1,\ 3/2, \ldots$.
- We will use quantum fields that transform under definite representations of Poincaré. This means that as a quantum operator

$$U^\dagger(\Lambda, a)\phi_A(x)U(\Lambda, a) = M_A{}^B(\Lambda)\,\phi_B(\Lambda^{-1}x - a)\ , \tag{107}$$

  where the matrices $M_A{}^B(\Lambda)$ form a (possibly non-unitary) representation of Lorentz.
- Representations of Poincaré in terms of the U operators acting on states is more complicated because they have to be unitary. Any representation can be characterized by the values of the Casimir operators P^2 and W^2 that commute with all the generators. It turns out that for momentum eigenstates, we only need to find representations of the much simpler little group. For massive states, $P^2 = M^2 > 0$, the little group is $SU(2)$ and they can be labelled by their momentum and their spin. For massless states, we have a slightly different little group $ISO(2)$ and the states are labelled by momentum and helicity.

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