PHYS 526 Notes #3: Interacting Scalars

David Morrissey
November 18, 2013

We have found that quantizing a scalar field theory with $V(\phi) = \frac{1}{2}m^2\phi^2 + \Lambda$ produces a theory of identical free particles with mass m, in which the individual particles do not interact with each other at all. Due to the simplicity of this theory, we were able to find all the energy eigenstates and eigenvalues. On the other hand, this simplicity also means that the theory is not very interesting.

In this note, we will look at more complicated scalar theories with higher-order terms in the potential. Specifically, we will investigate the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \Lambda - \Delta V(\phi) \ . \tag{1}$$

In many cases, we will take

$$\Delta V = \frac{\lambda}{4!}\,\phi^4 \ , \tag{2}$$

where λ is a constant parameter. For this choice of ΔV, the classical equation of motion is

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\,\phi^3 \ . \tag{3}$$

This is a non-linear partial differential equation. It is much harder to solve than a linear equation because it is no longer always true that a linear combination of two individual solutions is also solution. In general, we don't know how to solve such equations analytically.

Turning to the Hamiltonian formulation of the classical theory, we find the conjugate momentum

$$\Pi(x) = \partial_t\phi(x) \ , \tag{4}$$

and the Hamiltonian

$$\begin{aligned} H &= \int d^3x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \Lambda\right] + \int d^3x\ \Delta V(\phi) & (5)\\ &:= \bar{H}_0 + \Delta H \ , & (6) \end{aligned}$$

where $\Delta H = \int d^3x\, \Delta V(\phi)$.

1 Quantizing

To quantize this theory, we proceed just like before by elevating $\phi(x)$ and $\Pi(x)$ to operators on a Hilbert space and imposing canonical commutation relations on them at equal times. Also as before, we can define $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators at the fixed reference time $t_0 = 0$ according to

$$a(\vec{k}) = i\int d^3\, e^{ik\cdot x}(\Pi - ik^0\phi)|_{t=0}\ , \tag{7}$$

$$a^\dagger(\vec{k}) = -i\int d^3\, e^{-ik\cdot x}(\Pi + ik^0\phi)|_{t=0}\ , \tag{8}$$

where $k^0 = E_k = \sqrt{\vec{k}^2 + m^2}$. Using the commutators of ϕ and Π at $t = 0$, the commutators of the a and $a^\dagger$ operators are

$$[a(\vec{k}), a^\dagger(\vec{p})] = (2\pi)^3 2k^0\delta^{(3)}(\vec{k} - \vec{p}), \quad [a(\vec{k}), a(\vec{p})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{p})]\ . \tag{9}$$

Building up the Hamiltonian at $t = 0$ in terms of the modes, we find

$$H = \int \widetilde{dk}\, k^0 a^\dagger(\vec{k})a(\vec{k}) + \int d^3x\ \Delta V(\phi) \tag{10}$$

$$= H_0 + \Delta H(t = 0)\ . \tag{11}$$

Even though we have constructed the Hamiltonian from quantities defined at $t = 0$, it is still time-independent,[1] and so this expression holds for any time: $H(t) = H(0)$. We can also expand $\Delta H(0)$ in terms of the ladder operators, but the resulting expression will typically be complicated. For $\Delta H = \lambda\phi^4/4!$, it will involve products of four a and $a^\dagger$ factors.

Using the a and $a^\dagger$ operators, we can also build the Hilbert space. As before we assume there exists a state $|0\rangle$ that is annihilated by all the $a(\vec{k})$. In particular, $H_0|0\rangle = 0$. A complete basis of eigenstates of the Hermitian operator H_0 can then be formed by applying powers of $a^\dagger(\vec{p}_i)$ to $|0\rangle$. This is good – we have now quantized the theory by specifying the degrees of freedom and finding a basis for the corresponding Hilbert space.

Now for the bad news. For just about any ΔV that isn't linear or quadratic in ϕ, we do not know how to find the exact time evolution of the system. This difficulty is easiest to see in the Schrödinger picture, where operators are time-independent and the states evolve in time. If we try to identify $|0\rangle$ with a zero-particle state at $t = 0$, as time goes by it will evolve to $|0(t)\rangle = e^{-iHt}|0\rangle$. Since $H = (H_0 + \Delta H)$, and the ΔH piece contains stuff like $a[a^\dagger]^3$ (for $\lambda\phi^4/4!$), the time-evolved "vacuum" state will contain multi-particle components.

We can also see the challenge in the Heisenberg picture. Starting from $a(\vec{k})$ defined at $t = 0$, we can evolve it forward in time in the usual way:

$$a(t, \vec{k}) := e^{iHt}a(\vec{k})e^{-iHt}\ , \tag{12}$$

[1] This is just one of the basic assumptions of quantum mechanics for closed systems.

and similarly for $a^\dagger$. In the free theory, we had the very special result that $a(t,\vec{k}) = e^{-ik^0 t}a(\vec{k})$; time evolution only changed the mode operators by a phase. Unfortunately, this simple result no longer holds in the interacting theory, and $a(t,\vec{k})$ will be a complicated linear combination of products of multiple $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators. In general, we do not how to find the energy eigenstates of the full Hamiltonian when $\Delta V \neq 0$. Whatever shall we do?

2 Perturbing Around the Free Theory

The way out of this pit of despair is to give up on trying to compute the energy eigenstates exactly, and content ourselves with approximations to the truth. To do this, we will start with the free theory ($\Delta H = 0$), and expand around it in powers of ΔH. Such an expansion will be useful if ΔH is in some sense small relative to H_0. We will also focus on the relatively easy problem of scattering, where particles come in from spatial infinity at $t \to -\infty$, scatter with each other, and travel off to spatial infinity at $t \to +\infty$. This lends itself well to expanding around the free theory because particles separated by large distances, as they are before and after scattering, are expected to behave like free particles. On a more technical side, we will show in `notes-04` that the only quantities needed to compute scattering amplitudes are the *n-point* functions $\langle\Omega|T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\}|\Omega\rangle$, where $|\Omega\rangle$ is the vacuum state of the full theory.

The first step in perturbation theory is to construct a set of eigenstates of H_0. Note that even though $H_0 \neq H$, it is still a Hermitian operator and its eigenstates will therefore be a complete and orthonormalizable set. In fact, we have already done this: the lowest state is $|0\rangle$ all the higher states that can be built from it by applying factors of $a^\dagger(\vec{p})$.

In expanding around these H_0 eigenstates, we will make two physically reasonable assumptions. They can justified when the interaction ΔH is sufficiently small.[2] They are:

1. There exists a unique, normalizable, lowest-energy ground state $|\Omega\rangle$ of the full Hamiltonian H with no 3-momentum (*i.e.* $\vec{P}|\Omega\rangle = 0$) and non-zero overlap with $|0\rangle$:
$$\langle 0|\Omega\rangle \neq 0 \ . \tag{13}$$
Note that in general, $|0\rangle$ is different from $|\Omega\rangle$.

2. The next state in the spectrum is an isolated one-particle state with momentum $\vec{p}$ and energy $E = \sqrt{\vec{p}^2 + M^2}$ for some mass M (possibly different from m). By isolated, we mean that there is a non-zero energy gap above $|\Omega\rangle$, and another energy gap between the one-particle state (in the Lorentz frame with $\vec{p} = \vec{0}$) and the next set of states.

3. The field has vacuum expectation value of zero, $\langle\Omega|\phi(0)|\Omega\rangle = 0$.

The first assumption means that the interaction does not change the vacuum of the theory in too radical a way. The second and third imply that we will still be able to associate the field $\phi(x)$ with a specific particle species.

[2]However, they can be explicitly violated at *strong coupling*, when ΔH cannot be treated as small.

2.1 The Interaction Picture

To expand the full theory around the free theory, let us begin by writing $\phi(0,\vec{x})$ in terms of the ladder operators by inverting Eqs. (7,8):

$$\phi(0,\vec{x}) = \int \widetilde{dk} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right] \ . \tag{14}$$

To evolve the field to later times, we use the full Hamiltonian:

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \ . \tag{15}$$

Unfortunately, we don't know how to simplify this because we don't know how to move H through the $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators that make up $\phi(0,\vec{x})$.

In the face of this challenge, let us do something easier and define the time-dependent *interaction picture* field by

$$\begin{aligned} \phi_I(t,\vec{x}) &:= e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} & (16)\\ &= \int \widetilde{dk} \ \left[a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right] \ , & (17) \end{aligned}$$

where H_0 is the free Hamiltonian at $t=0$. Here, the time evolution is simple because we do know how to commute H_0 with a and $a^\dagger$. We can also generalize this definition to any other local operator: $\mathcal{O}_I(t) := e^{iH_0t}\mathcal{O}(0)e^{-iH_0t}$.

Before moving on, let us briefly take note of the time dependence of the full Hamiltonian and its pieces. We have

$$\begin{aligned} H(t) &= e^{iHt}H(0)e^{-iHt} \ = \ H(0) & (18)\\ &= e^{iHt}H_0e^{-iHt} + e^{iHt}\Delta H(0)e^{-iHt} & (19)\\ &:= \tilde{H}_0(t) + \Delta H(t) \ . & (20) \end{aligned}$$

While the full Hamiltonian is time-independent (since it commutes with itself), the terms within it need not be. We have defined here a time-dependent version $\tilde{H}_0(t)$ of the free Hamiltonian such that $H_0 = \tilde{H}_0(0)$. The tilde is to distinguish it from H_0, which is given by the explicit expression in Eq. (10).

The interaction picture is useful because it factors out the time evolution due to the free Hamiltonian. We can relate any Heisenberg-picture operator to the interaction picture version by

$$\mathcal{O}(t) = U_I^\dagger(t)\mathcal{O}_I(t)U_I(t) \ , \tag{21}$$

where

$$U_I(t) = e^{iH_0t}e^{-iHt} \ . \tag{22}$$

Differentiating with respect to t, this implies

$$i\partial_t U_I(t) = \Delta H_I(t) U_I(t) \ , \tag{23}$$

where $\Delta H_I(t) = e^{iH_0 t}\Delta H(0)e^{-iH_0 t}$ is the interaction Hamiltonian in the interaction picture. Note that if $\Delta H = \int d^3x\, \lambda^{(m,n)}\Pi^m\phi^n$, we have $\Delta H_I = \int d^3x\, \lambda^{(m,n)}\Pi_I^m\phi_I^n$.

The solution to Eq. (23) for $t > 0$ is derived in detail in Peskin & Schroeder [1], and is given by *Dyson's equation*:

$$U_I(t) = T\left\{\exp\left[-i\int_0^t dt'\Delta H_I(t')\right]\right\} \tag{24}$$

$$= \mathbb{I} + \frac{(-i)}{1!}\int_0^t dt_1\,\Delta H_I(t_1) + \frac{(-i)^2}{2!}\int_0^t dt_1\int_0^t dt_2\, T\{\Delta H_I(t_1)\,\Delta H_I(t_2)\} + \ldots \tag{25}$$

where the first line is just a compact shorthand for the second. This expression certainly has the right boundary condition, and it is straightforward to check that it satisfies Eq. (23) by explicit differentiation. For $t < 0$, the solution is

$$U_I(t) = T'\left\{\exp\left[-i\int_0^t dt'\Delta H_I(t')\right]\right\}\ , \tag{26}$$

where T' denotes reverse time ordering

$$T'\{\phi(x_1)\phi(x_2)\} = \Theta(t_1 - t_2)\,\phi(x_2)\phi(x_1) + \Theta(t_2 - t_1)\,\phi(x_1)\phi(x_2)\ . \tag{27}$$

To combine both cases into a simple notation, let us define $\widetilde{T}$ to be time ordering for $t > 0$ and reverse time ordering for $t < 0$. Thus,

$$U_I(t) = \widetilde{T}\left\{\exp\left[-i\int_0^t dt'\Delta H_I(t')\right]\right\}\ . \tag{28}$$

In terms of $U(t)$, we now have an analytic (but very complicated) relation between $\phi(t,\vec{x})$ and $\phi_I(t,\vec{x})$.

We can generalize $U(t)$ by defining

$$U_I(t_2,t_1) = U_I(t_2)U_I^\dagger(t_1) = e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1}\ . \tag{29}$$

It can be shown that this quantity is equal to

$$U_I(t_2,t_1) = \widetilde{T}\left\{\exp\left[-i\int_{t_1}^{t_2} dt'\Delta H_I(t')\right]\right\}\ , \tag{30}$$

where $\widetilde{T}$ is time ordering for $t_2 > t_1$ and reverse time-ordering for $t_2 < t_1$.

The $U_I(t_1,t_2)$ operator has some very nice properties. They include:

1. $U_I(t_1,t_1) = \mathbb{I}$
2. $U_I(t_3,t_2)U_I(t_2,t_1) = U_I(t_3,t_1)$
3. $U_I^\dagger(t_2,t_1) = U_I^{-1}(t_2,t_1) = U_I(t_1,t_2)$

By definition, $U_I(t,0) = U_I(t)$. This implies trivially that $\phi(x) = U_I^\dagger(t,0)\phi_I(x)U_I(t,0)$. Thus, for any product of fields (where $\phi_i = \phi(x_i)$):

$$\begin{aligned}\phi_n\phi_{n-1}\ldots\phi_1 &= U_I^\dagger(t_n,0)\phi_{I_n}U_I(t_n,0)U_I^\dagger(t_{n-1},0)\phi_{I_{n-1}}\ldots U_I(t_2,0)U_I^\dagger(t_1,0)\phi_{I_1}U_I(t_1,0) \\ &= U_I^\dagger(t_n,0)\,\phi_{I_n}\,U_I(t_n,t_{n-1})\,\phi_{I_{n-1}}\ldots\phi_{I_2}\,U_I(t_2,t_1)\,\phi_{I_1}\,U_I(t_1,0)\ .\end{aligned} \tag{31}$$

In this expression, the U_I factors can be thought of as transfer operators that connect the interaction-picture operators at different times.

2.2 To Infinity and Beyond

Our goal is to compute the time-ordered operator expectation values $\langle\Omega|T\{\phi_n,\ldots\phi_1\}|\Omega\rangle$. Our strategy to do so will be to rewrite them in terms of interaction picture fields sandwiched between the H_0 vacuum $|0\rangle$. This will give us something we know how to deal with, since interaction picture fields act on the $|0\rangle$ state exactly like in the free theory. Evaluating them is then just a matter of commuting a bunch of a and $a^\dagger$ operators through each other.

As a first step, let us expand the H_0 eigenstate $|0\rangle$ in terms of energy eigenstates $\{|N\rangle\}$ of the full Hamiltonian H in a clever way. For this, recall that $H_0|0\rangle = 0$ by construction, and let us write E_Ω as the energy of the true vacuum: $H|\Omega\rangle = E_\Omega|\Omega\rangle$. Multiplying $|0\rangle$ by $U_I^{-1}(-\tau,0)$, we get

$$U_I^{-1}(-\tau,0)|0\rangle = e^{-iH\tau}e^{iH_0\tau}|0\rangle \tag{32}$$

$$= e^{-iH\tau}\left(|\Omega\rangle\langle\Omega| + \sum_{N>\Omega}|N\rangle\langle N|\right)e^{iH_0\tau}|0\rangle \tag{33}$$

$$= e^{-iE_\Omega\tau}\langle\Omega|0\rangle\ |\Omega\rangle + \sum_{N>\Omega}e^{-iE_N\tau}\langle N|0\rangle\ |N\rangle\ , \tag{34}$$

where in the second line we have inserted unity in the form of a complete set of H eigenstates. This relation simplifies enormously in the limit $\tau \to \infty(1-i\epsilon)$, with $0 < \epsilon \ll 1$. Since $E_N > E_\Omega$, the contributions from all the higher modes become exponentially suppressed in this limit. It follows that

$$|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)}\frac{e^{iE_\Omega\tau}}{\langle\Omega|0\rangle}\,U_I^{-1}(-\tau,0)|0\rangle\ . \tag{35}$$

A similar argument gives

$$\langle\Omega| = \lim_{\tau\to\infty(1-i\epsilon)}\frac{e^{iE_\Omega\tau}}{\langle 0|\Omega\rangle}\,\langle 0|U_I(\tau,0)\ . \tag{36}$$

The second step is to combine these results with the expression of Eq. (31). This gives

$$\langle\Omega|\phi_n\ldots\phi_1|\Omega\rangle = \lim_{\tau\to\infty 1-i\epsilon}\frac{e^{2iE_\Omega\tau}}{|\langle\Omega|0\rangle|^2}\,\langle 0|U_I(\tau,0)U_I^\dagger(t_n,0)\phi_{I_n}\ldots\phi_{I_1}U_I(t_1,0)U_I^{-1}(-\tau,0)|0\rangle\ . \tag{37}$$

Making use of the properties of the $U_I(t,t')$ operators, it is straightforward to show that

$$U_I(\tau,0)U_I^\dagger(t_n,0) = U_I(\tau,t_n)\ , \qquad U_I(t_1,0)U_I^{-1}(-\tau,0) = U_I(t_1,-\tau^*)\ . \tag{38}$$

Thus, we have

$$\langle\Omega|\phi_n\dots\phi_1|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)} \frac{e^{2iE_\Omega\tau}}{|\langle\Omega|0\rangle|^2}\,\langle 0|U_I(\tau,t_n)\phi_{I_n}U_I(t_n,t_{n-1})\dots\phi_{I_1}U_I(t_1,-\tau^*)|0\rangle\ . \tag{39}$$

Everything has contracted nicely.

The last two steps are to deal with the funny prefactor and to focus on time-ordered products of fields. For the prefactor, the normalization condition on $|\Omega\rangle$ implies

$$\begin{aligned}
1 &= \langle\Omega|\Omega\rangle && (40)\\
&= \lim_{\tau\to\infty(1-i\epsilon)} \frac{e^{2iE_\Omega\tau}}{|\langle\Omega|0\rangle|^2}\,\langle 0|U_I(\tau,0)U_I^{-1}(-\tau,0)|0\rangle && (41)\\
&= \lim_{\tau\to\infty(1-i\epsilon)} \frac{e^{2iE_\Omega\tau}}{|\langle\Omega|0\rangle|^2}\,\langle 0|U_I(\tau,-\tau^*)|0\rangle && (42)\\
&= \lim_{\tau\to\infty(1-i\epsilon)} \frac{e^{2iE_\Omega\tau}}{|\langle\Omega|0\rangle|^2}\,\langle 0|T\left\{\exp\left[-i\int_{-\tau}^{\tau}dt'\,\Delta H_I(t')\right]\right\}|0\rangle && (43)
\end{aligned}$$

In the last line, we have just applied the solution for $U_I(t,t')$ given in Eq. (30).

We can also apply our solution for $U(t,t')$ to the operator in Eq. (39). If not for the insertions of ϕ_{I_i} in this operator, we could contract up all the $U(t_i,t_{i-1})$ factors to give a single $U(\tau,-\tau^*)$ factor. Unfortunately, ϕ_{I_i} and $\Delta H_I(t)$ typically do not commute and such a simplication is not possible in general. However, we can make this simplication within a time-ordering symbol, since the ordering of operators inside the symbol does not matter.[3] Therefore, we can write

$$\langle\Omega|T\{\phi_n\dots\phi_1\}|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_{I_1}\dots\phi_{I_n}\exp\left[-i\int_{-\tau}^{\tau}dt'\Delta H_I(t')\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-\tau}^{\tau}dt'\Delta H_I(t')\right]\right\}|0\rangle}\ . \tag{44}$$

This result will be used a lot, and we will sometimes call it our *master formula*.

We will show in `notes-04` that time-ordered vacuum matrix elements of fields are precisely what are needed to compute scattering rates. Since they come up so much, they are given a special name and an abbreviated notation. The time-ordered product of n fields is called the n-point function $G^{(n)}(x_1,\dots,x_n)$, and is often written as

$$\langle\Omega|T\{\phi_1\dots\phi_n\}|\Omega\rangle = G^{(n)}(x_1,\dots,x_n) = \langle\phi_1\dots\phi_n\rangle\ . \tag{45}$$

Onward!

[3] After all, their order will be completely determined by the time ordering, *e.g.* $T\{\phi_1\phi_2\} = T\{\phi_2\phi_1\}$.

2.3 Computing Stuff, Finally

After all this formalism, let's actually compute something. The way the exact result of Eq. (44) is usually too complicated to compute exactly. Instead, it usually computed perturbatively by expanding it in powers of ΔH_I. The right-hand side of Eq. (44) is written in terms of interaction-picture fields and the H_0 vacuum $|0\rangle$. This is nice because the interaction picture fields have exactly the same expansion in terms of the a and $a^\dagger$ ladder operators as a free field, Eq. (17). Since we know how these act on $|0\rangle$ and how they commute with each other, we know how to compute all the matrix elements.

To be concrete, let us specialize to the case of

$$\Delta H = \int d^3x \, \frac{g}{3!}\phi^3 \ . \tag{46}$$

It follows that $\Delta H_I(t) = \int d^3x \frac{g}{3!}\phi_I^3(t,\vec{x})$. With this in mind, we will evaluate Eq. (44) as an expansion in the coupling g.

The easiest thing to compute is the expectation value of the identity operator. For this, Eq. (44) simply gives

$$\langle \mathbb{I} \rangle = 1 \ . \tag{47}$$

Not so bad at all.

A slightly more challenging quantity is the 2-point function,

$$\langle \phi_1 \phi_2 \rangle = \frac{\langle 0 | T\left\{ \phi_{I_1}\phi_{I_2} \exp\left[-i \int d^4z \, \frac{g}{3!}\phi_I^3(z) \right] \right\} |0\rangle}{\langle 0 | T\left\{ \exp\left[-i \int d^4z \, \frac{g}{3!}\phi_I^3(z) \right] \right\} |0\rangle} \ . \tag{48}$$

To leading non-trivial order, which is g^0 in this case, the numerator is

$$\langle 0 | T\{\phi_I(x_1)\phi_I(x_2)\} |0\rangle = D_F(x_1 - x_2) \ , \tag{49}$$

while the denominator is equal to unity. There are higher-order contributions, but it is reassuring that we reproduce the free theory result at lowest order.

A quantity that only arises at order g^1 in this theory is the 3-point function. Evaluating it at this order,

$$\begin{aligned} G^{(3)}(x_1,x_2,x_3) &= \langle 0 | T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3) \left[1 - i \int d^4z \frac{g}{3!}\phi_I^3(z) \right] |0\rangle + \dots & (50) \\ &= 0 + \frac{(-i)g}{3!} \int d^4z \, \langle 0 | T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I^3(z)\} |0\rangle + \dots & (51) \end{aligned}$$

The terms omitted in each line are higher-order in g (g^2 or higher). In particular, at this order we can set the denominator to unity. The second line can be evaluated by expanding the fields in terms of ladder operators, arranging their time ordering, and commuting the ladder operators to put everything in normal order. This requires a lot of brute-force calculation to do in general, but fortunately there is a systematic way of computing them called Wick's Theorem.

3 Wick's Theorem

To compute the vacuum matrix elements of time-ordered products of interacting fields, Eq. (44) tells us we need to manipulate products of a and $a^\dagger$ operators sandwiched between $\langle 0|$ and $|0\rangle$. This is straightforward to do in principle, but the many different terms that arise make it very tedious in practice. *Wick's Theorem* is way to reduce the amount of work.

Wick's Theorem is a statement about free fields that also applies to fields in the interaction picture. In words, it is

$$T\left\{\phi(x_1)\dots\phi(x_n)\right\} = N\left\{\phi(x_1)\dots\phi(x_n) + \text{all contractions}\right\} \ . \tag{52}$$

We will explain what we mean by a contraction below, but for now let us just say that it takes two field operators and turns them into a number. The power of this formula is that a normal-ordered operator vanishes when sandwiched between $\langle 0|$ and $|0\rangle$ unless it is proportional to the identity. This means that only the terms on the right-hand side that are fully contracted, such that no field operators are left, can contribute to the vacuum expectation value.

To describe Wick's theorem, it will be useful to split up the free-field operator into two pieces:

$$\phi(x) = \int \widetilde{dk}\, \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] = \phi_-(x) + \phi_+(x) \ , \tag{53}$$

with[4]

$$\phi_-(x) = \int \widetilde{dk}\, a(\vec{k})e^{-ik\cdot x}, \qquad \phi_+(x) = \int \widetilde{dk}\, a^\dagger(\vec{k})e^{ik\cdot x} \ . \tag{54}$$

From this definition, we find that

$$[\phi_-(x_1), \phi_-(x_2)] = 0 = [\phi_+(x_1), \phi_+(x_2)] \ , \tag{55}$$

together with

$$\phi_-^\dagger(x) = \phi_+(x) \ , \tag{56}$$

as well as

$$[\phi_-(x_1), \phi_+(x_2)] = \int \widetilde{dk}\, e^{-ik\cdot(x_1-x_2)} := D(x_1 - x_2) \ . \tag{57}$$

In terms of ϕ_- and ϕ_+, a product of fields is normal-ordered if and only if all the ϕ_- pieces lie to the right of all the ϕ_+ pieces.

[4]Note that my + and − are flipped relative to P&S.

Since $T\{\mathbb{I}\} = N\{\mathbb{I}\}$ and $T\{\phi(x)\} = N\{\phi(x)\}$ are both trivial, let's look at the product of two fields:

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\} &= \Theta(t_1-t_2)\,[\phi_-(x_1)+\phi_+(x_1)]\,[\phi_-(x_2)+\phi_+(x_2)] && (58)\\ &\quad +\Theta(t_2-t_1)\,[\phi_-(x_2)+\phi_+(x_2)]\,[\phi_-(x_1)+\phi_+(x_1)] \\ &= \Theta_{12}\,(\phi_{1-}\phi_{2-}+\phi_{1+}\phi_{2+}+\phi_{1+}\phi_{2-}+\phi_{2+}\phi_{1-}+[\phi_{1-},\phi_{2+}]) && (59)\\ &\quad +\ \Theta_{21}\,(\phi_{2-}\phi_{1-}+\phi_{2+}\phi_{1+}+\phi_{2+}\phi_{1-}+\phi_{1+}\phi_{2-}+[\phi_{2-},\phi_{1+}]) \\ &= \phi_{1-}\phi_{2-}+\phi_{1+}\phi_{2-}+\phi_{2+}\phi_{1-}+\phi_{1+}\phi_{2+} && (60)\\ &\quad +\ \Theta(t_1-t_2)D(x_1-x_2)+\Theta(t_2-t_1)D(x_2-x_1) \\ &= N\{\phi_1\phi_2\}+D_F(x_1-x_2)\ . && (61) \end{aligned}$$

In the third line, we have used $\Theta_{12}+\Theta_{21}=1$, while in the fourth we have combined the two D functions into a D_F.[5] Let us now define the contraction of two fields to be

$$\overbrace{\phi(x_1)\phi(x_2)} = D_F(x_1-x_2)\ . \tag{62}$$

With this in place, we have

$$T\{\phi(x_1)\phi(x_2)\} = N\{\phi(x_1)\phi(x_2)+\overbrace{\phi(x_1)\phi(x_2)}\}\ . \tag{63}$$

Note that since D_F is just a function proportional to the unit operator, we can freely move it inside the normal ordering. This is consistent with our statement of Wick's Theorem.

Doing the same calculation for three fields, one obtains

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\} &= N\{\phi_1\phi_2\phi_3+\phi_1\overbrace{\phi_2\phi_3}+\phi_2\overbrace{\phi_3\phi_1}+\phi_3\overbrace{\phi_1\phi_2}\} && (64)\\ &= N\{\phi_1\phi_2\phi_3+\phi_1 D_F(x_2-x_3)+\phi_2\,D_F(x_1-x_3)+\phi_3\,D_F(x_1-x_2)\}. && (65) \end{aligned}$$

For four fields,

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} &= N\{\phi_1\phi_2\phi_3\phi_4+\overbrace{\phi_1\phi_2}\,\phi_3\phi_4+\overbrace{\phi_1\phi_3}\,\phi_2\phi_4+\overbrace{\phi_1\phi_4}\,\phi_2\phi_3 \\ &\quad +\overbrace{\phi_2\phi_3}\,\phi_1\phi_4+\overbrace{\phi_2\phi_4}\,\phi_1\phi_3+\overbrace{\phi_3\phi_4}\,\phi_1\phi_2 \\ &\quad +\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}+\overbrace{\phi_1\phi_3}\,\overbrace{\phi_2\phi_4}+\overbrace{\phi_1\phi_4}\,\overbrace{\phi_2\phi_3}\}\ . \end{aligned} \tag{66}$$

Oof.

The proof of Wicks' theorem goes by induction. Suppose it holds for any product of n fields. For $(n+1)$ fields labelled such that $t_1 > t_2 > \ldots$, we get

$$\begin{aligned} T\{\phi_1\phi_2\ldots\phi_{n+1}\} &= \phi_1 T\{\phi_2\ldots\phi_{n+1}\} && (67)\\ &= (\phi_{1+}+\phi_{1-})\,N\{\phi_2\ldots\phi_{n+1}+\text{contractions}'\} && (68) \end{aligned}$$

where contractions$'$ means all contractions that do not involve ϕ_1. In this expression, the ϕ_{1+} term is already in normal order, so we only need to move ϕ_{1-} through to the right. This will

[5] Check this for yourself by expanding out the fields in the definition of D_F.

eventually produce something that is normal-ordered together with a bunch of contractions, noting that $[\phi_-(x_1), \phi_+(x_i)] = D_F(x_1 - x_i)$ since we have $t_1 > t_i$ for all i. All that remains to show is that all possible contractions involving ϕ_1 are produced. We can do this by giving an algorithm to build a given contraction given ϕ_{1-} on the left and the terms already assumed to be present in the inductive step. I won't do this explicitly, but it isn't too difficult with a bit of fiddling.

Our statement and (partial) proof of Wick's Theorem was for free scalar fields. However, it also applies to scalar fields in the interaction picture since they obey the same commutation relations and have the same ladder operator expansions as free fields. Therefore Wick's theorem allows us to systematically evaluate the operators that arise in the vacuum expectation values of time-ordered products of fields.

4 A First Look at Feynman Diagrams and Rules

The contractions that arise in computing n-point functions can be represented by diagrams. These *Feynman diagrams* help to keep track of all the possible terms. Even better, it is possible to assign a value to each Feynman diagram. Computing n-point functions can thus be reduced to drawing a set of pictures and computing their values using a set of *Feynman rules*.

To illustrate how Feynman diagrams work, let us return to the interacting scalar theory with $\Delta V = g\phi^3/3!$ and evaluate the leading-order expression for the 3-point function we found in Eqs. (50,51). Applying Wick's Theorem,

$$\begin{aligned}\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\,\phi_I^3(z)\}|0\rangle \;\; = \qquad\qquad (69)\\ (3!)D_F(x_1-z)D_F(x_2-z)D_F(x_3-z) + 3\,D_F(x_1-z)D_F(x_2-x_3)D_F(z-z)\\ +\;\; 3\,D_F(x_2-z)D_F(x_1-x_3)D_F(z-z) \;\; + \;\; 3\,D_F(x_3-z)D_F(x_1-x_2)D_F(z-z)\end{aligned}$$

Note that only fully contracted terms in the Wick expansion contribute to the matrix element. The numerical prefactors are just the number of ways to get each specific contraction.

We can represent this set of contractions in pictures. To each term, we associate a distinct Feynman diagram. The set of diagrams corresponding to Eq. (69) is shown in Fig. 1. The rules for drawing a diagram are as follows:

1. For each coordinate x_i, draw a dot.
2. A contraction producing a factor of $D_F(x_i - x_j)$ is represented by a line connecting the dot for x_j to the dot for x_i.
3. Each factor of $D_F(z - z)$ is depicted as a loop connecting z to itself.

The utility of these rules is that we can invert them, using diagrams to figure out all the possible contractions, and then associating a number to each diagram. Instead of wading through a mire of raising and lowering operators, we only need to connect up some dots.

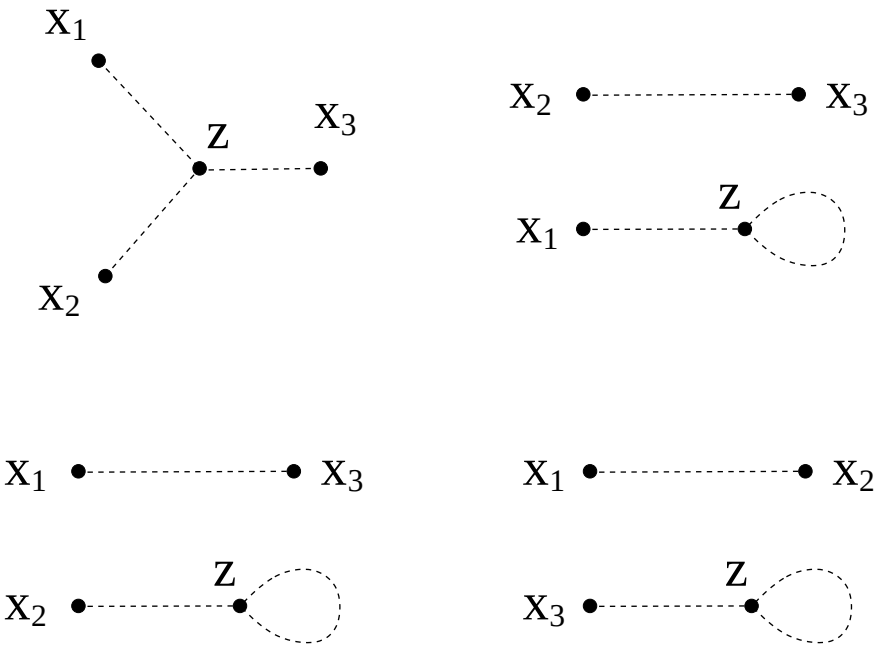

Figure 1: Feynman diagrams for the 3-point function at $\mathcal{O}(g)$ of Eq. (69).

Let's now build up a set of *Feynman rules* for computing time-ordered vacuum matrix elements of fields in the $g\phi^3/3!$ theory. These rules will specify how to draw a set of Feynman diagrams for a given matrix element and assign a numerical value to each diagram. They are based on the master formula of Eq. (44), which we treat by expanding the exponentials it contains in a power series in g up to some fixed order g^M. At this order, the perturbation theory estimate for the matrix element is given by the sum of each of the individual g^0, $g^1, \ldots, g^M$ contributions. If g is small enough, keeping only the first few terms in the expansion should give a good approximation to the truth.

The Feynman rules to compute the g^M contribution to the n-point function $G^{(n)}(x_1, \ldots, x_n) = \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle$ in the $g\phi^3/3!$ theory are:

1. Draw a dot for each x_i coordinate. We call these *external points*.

2. Draw another M dots and associate a coordinate z_j $(j = 1, 2, \ldots M)$ to each of them. We call these *vertices*.

3. Draw Feynman diagrams by connecting the dots with lines in all possible ways following two simple rules:
   a) Each external point has a single line connected to it.
   b) Each vertex has three lines connected to it.

4. Remove all the diagrams in which there is a *vacuum bubble* – a vertex point that not connected by any path to any of the external points.

5. Write the number corresponding to each Feynman diagram:
   a) For every vertex with coordinate z_j, write a factor of $-ig\int d^4z_j$.

b) For every line connecting points a and b (external or vertex),
add a factor of the Feynman propagator $D_F(a-b)$.
c) Any diagram with an unconnected dot or an unpaired line is equal to zero.
d) Multiply the result for each diagram by $1/M!$ and a *symmetry factor*.

The net g^M contribution is then the sum of all the diagrams.

To see how this works in practice, let us begin with the 2-point function and work to second order (g^2) in the perturbative expansion. From our master formula, we have

$$\begin{aligned} G^{(2)}(x_1,x_2) &= \langle 0|T\left\{\phi_1\phi_2\left[1+\frac{(-i)g}{3!}\int d^4z\,\phi_z^3+\frac{(-i)^2g^2}{3!3!2!}\int d^4z_1\int d^4z_2\,\phi_{z_1}^3\phi_{z_2}^3\right]\right\}|0\rangle \\ &\Big/\langle 0|T\left\{1+\frac{(-i)g}{3!}\int d^4z\,\phi_z^3+\frac{(-i)^2g^2}{3!3!2!}\int d^4z_1\int d^4z_2\,\phi_{z_1}^3\phi_{z_2}^3\right\}|0\rangle \end{aligned} \tag{70}$$

Following the rules and drawing the Feynman diagrams, we find the set shown in Fig. 2. At leading order g^0, we only have the first diagram in the figure and associate to it the number

$$G^{(2)}(x_1,x_2)|_{g^0} = D_F(x_1-x_2)\ . \tag{71}$$

This matches what we would have found from applying Wick's Theorem to the matrix element. There are no contributions proportional to g^1 because it is impossible to connect up all the dots and all the lines. Again, this matches what the matrix element would produce. Going to g^2, we find the second row of diagrams in Fig. 2. Their sum is

$$\begin{aligned} G^{(2)}(x_1,x_2)|_{g^2} &= \\ &\left[\frac{1}{2}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\int d^4z_2\ D_F(x_1-z_1)D_F(z_1-z_2)D_F(z_1-z_2)D_F(z_2-x_2) \\ &+\left[\frac{1}{2}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\int d^4z_2\,D_F(x_1-z_1)D_F(z_1-z_2)D_F(z_2-z_2)D_F(z_1-x_2) \\ &+\left[\frac{1}{4}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\,D_F(x_1-z_1)D_F(z_1-z_1)\int d^4z_2\,D_F(x_2-z_2)D_F(z_2-z_2) \end{aligned} \tag{72}$$

The factors of $(2!/2!)$ come from the $(1/2!)$ in the expansion of the exponential together with the fact that since z_1 and z_2 are integrated over, diagrams differing only by the permutation of z_1 and z_2 have the same value. The numerical prefactors in the square brackets $(1/2, 1/2, 1/4)$ are the symmetry factors. Note as well that we have omitted the lower row of diagrams in Fig. 2 (surrounded by dotted boxes) because they have a vacuum bubble, in which one or more vertices are not connected to any of the external points in any way. The full 2-point function in the interacting theory up to order g^2 in perturbation theory is therefore

$$G^{(2)}(x_1,x_2) = G^{(2)}(x_1,x_2)|_{g^0} + G^{(2)}(x_1,x_2)|_{g^2} + \dots\ . \tag{73}$$

This is just a number to be evaluated.

Most of the Feynman rules are easy to relate to what's going on in matrix element after applying Wick's Theorem. However, rules 4 and 5d) might seem a bit strange. For rule 4,

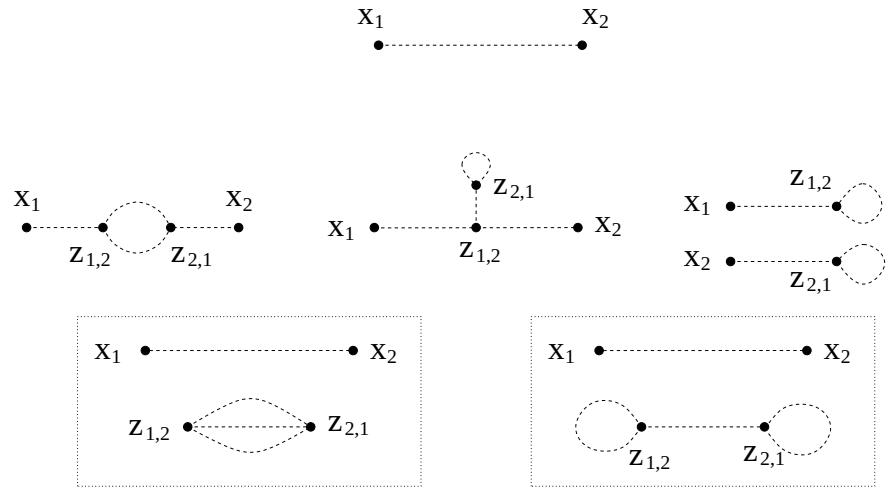

Figure 2: Feynman diagrams for the 2-point function at $\mathcal{O}(g^2)$.

the reason we can drop these diagrams is that they are cancelled off by the expansion of the denominator in Eq. (44). We can see this explicitly here by studying the matrix element. Applying Wick's theorem to the denominator in Eq. (70), we get

$$\begin{aligned} \text{Denom.} &= 1 + \left[\frac{1}{12}\right](-i)^2 g^2 \int d^4z_1 \int d^4z_2 \; [D_F(z_1 - z_2)]^3 \\ &\qquad + \left[\frac{1}{8}\right](-i)^2 g^2 \int d^4z_1 \int d^4z_2 \; D_F(z_1 - z_2) D_F(z_1 - z_1) D_F(z_2 - z_2) \\ &:= (1 + g^2\Delta) \; . \end{aligned} \tag{74}$$

Using $1/(1+g^2\Delta) = 1 - g^2\Delta + g^4\Delta^2 + \ldots$, the net result up to order g^2 is

$$G^{(2)}(x_1, x_2) = \big[D_F(x_1 - x_2) + g^2(\ldots)\big] \times (1 - g^2\Delta) \tag{75}$$

$$= D_F(x_1 - x_2)(1 - g^2\Delta) + g^2(\ldots) \; , \tag{76}$$

where the $g^2(\ldots)$ term corresponds to all the stuff in the numerator of the matrix element proportional to g^2. Some of the terms in the numerator will be the diagrams in the bottom row of Fig. 2 that contain vacuum bubbles. Using the Feynman rules, it is easy to show that these diagrams with vacuum bubbles sum to to $D_F(x_1 - x_2)\Delta$. These diagrams are therefore cancelled off at this order by the expansion of the denominator. It turns out that such a cancellation between the denominator and diagrams in the numerator containing vacuum bubbles is a general feature, and a proof can be found in Peskin & Schroeder [1]. This explains the origin of rule 4.

Some diagrams also require an additional symmetry factor. In many cases, summing over all the contractions cancels off the (1/3!) in our definition of $\Delta V = g\phi^3/3!$. However, sometimes this cancellation is incomplete and we need to correct for it with an additional factor, called the symmetry factor of the diagram. In general, it is equal to one divided by the number of ways to that internal lines can be reconnected to give the same diagram [1, 2].

As a practical matter, most people just work out the number of contractions corresponding to each diagram by counting contractions using Wick's theorem.

Finally, let us also mention that the denominator factor of 2! from the expansion of the exponential cancelled against the factor of 2! arising from the fact that for each diagram with fixed z_1 and z_2, there was a corresponding diagram with $z_1 \leftrightarrow z_2$ having the same numerical value. This is also a general feature at any order because all the internal z_i coordinates are integrated over. Thus, we can slightly modify our Feynman rule 5d) and leave out the $1/M!$ factor (at order g^M) with the understanding that we are only to include those diagrams that remain distinct when the z_i coordinates are permuted.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[2] M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*