PHYS 526 Homework #8

Due: Nov. 5, 2013

0. Read Chs.3,4.7-4.8 of Peskin & Schroeder.

1. Spinor Products

   a) Show that $(\sigma \cdot p)(\bar{\sigma} \cdot p) = p^2 = (\bar{\sigma} \cdot p)(\sigma \cdot p)$.

   b) Prove Eqs.(36 – 42) in `notes-07`.
   Hint: write out $\sigma \cdot p$ and $\bar{\sigma} \cdot p$ explicitly in terms of components.

2. Dirac spinor calculations.

   a) Write $[\bar{u}^a(p_2, s_2)u_a(p_1, s_1)]^*$ in terms of u's and $\bar{u}$'s (with no explicit u^* or $u^\dagger$ factors).
   Hint: $[(w^\dagger)^a M_a{}^b v_b]^* = [w^\dagger M v]^* = [w^\dagger M v]^\dagger = v^\dagger M^\dagger w = (v^\dagger)^a (M^\dagger)_a{}^b w_b$.

   b) Use this result to simplify $|\bar{u}^a(p_2, s_2)u_a(p_1, s_1)|^2$.
   Hint: be careful with dummy indices that are summed over:
   $|\bar{u}^a(p_2, s_2)u_a(p_1, s_1)|^2 = [\bar{u}^a(p_2, s_2)u_a(p_1, s_1)]^*[\bar{u}^b(p_2, s_2)u_b(p_1, s_1)]$.
   Don't use the same dummy index on both factors!

   c) Compute $\sum_{s_1} \sum_{s_2} |\bar{u}^a(p_2, s_2)u_a(p_1, s_1)|^2$ by using the completeness relation of Eq. (28) of `notes-07`.
   Hint: your final result should involve a trace over Dirac indices.

   d) Use the γ matrix trace tricks discussed in `notes-06` to evaluate the result of c).

3. The Dirac Hamiltonian.

   a) Show that $\int d^3x\, m\, \bar{\Psi}\Psi = \sum_s \int \widetilde{dk}\, \frac{m^2}{k^0} \left[a^\dagger(k,s)a(k,s) - b(k,s)b^\dagger(k,s)\right]$
   Hint: remember that $\int d^3x\ e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{p})$, *and use the spinor relations you found in #2.*

   b) Use the Dirac equation for Ψ to show that $-i\gamma^i\partial_i\Psi = (i\gamma^0\partial_0 - m)\Psi$.

   c) Show that $\int d^3x\ \bar{\Psi}(-i\gamma^i\partial_i)\Psi = \int \widetilde{dk}\, \frac{\vec{k}^2}{k^0} \left[a^\dagger(k,s)a(k,s) - b(k,s)b^\dagger(k,s)\right]$.

   d) Combine c) and a) to express the Dirac Hamiltonian in terms of the a and b mode operators.

4. In deriving the Hamiltonians for free scalars and fermions, we found in both cases that it was necessary to cancel off a formally infinite constant. We achieved this by adding a constant term to the Lagrangian density. Show that in a theory containing four real scalars ϕ_i, $i = 1, 2, 3, 4$, and a single Dirac fermion, Ψ, no such constant is needed provided all the particles have the same mass. This is precisely what happens in theories with *supersymmetry*, an extension of the Poincaré group that relates bosons and fermions. In particular, supersymmetry implies that the energy of the ground state is exactly zero.

5. Apply Noether's Theorem to derive the (classical) conserved corrent corresponding to the global $U(1)$ invariance of the free Dirac fermion. Reinterpret it as a quantum operator and rewrite the current as a normal-ordered expression in terms of the a and b mode operators.