PHYS 526 Homework #4

Due: Oct. 8, 2013

0. Read Chs. 4.4-4.6, 7.1-7.2 of Peskin & Schroeder, and Chs. 10-12 of Srednicki.

1. Do the inductive step for proving Wick's theorem for the specific case of $(n = 2) \to (n = 3)$. In other words, assume Eq. (63) is true, and use it to express $T\{\phi_1\phi_2\phi_3\}$ in terms of normal-ordered operators (*i.e.* Eq. (64)). You may take $t_1 > t_2 > t_3$.

2. Consider the real scalar theory with $\Delta V = \lambda\phi^4/4!$.

   a) Compute the *connected* 4-point function at order λ starting directly from the master formula. Use your result to formulate Feynman rules for this theory. In particular, what is the form of the fundamental vertex and what is its value?
   Hint: "connected" means that all the vertices and external points are connected to each other in some way.

   b) Use these Feynman rules to evaluate the full (*i.e.* connected and disconnected) 2-point function to order λ.

   c) Evaluate the connected part 4-point function to order λ^2.

   d) Evaluate the 89-point function to order λ^{17}.

   In all four cases, you can leave your result in terms of D_F functions.

3. In the real scalar theory with $\Delta V = g\phi^3/3!$, evaluate the 4-point function $G^{(4)}(x_1, x_2, x_3, x_4)$ up to order g^2 in perturbation theory. Draw the Feynman diagrams as well. You can leave your answer in terms of D_F functions. There are lots of terms, and you don't have to write out all of them. However, show at least one representative of each different type of term contributing to the full result.

4. A prelude to external states.
   Consider the "external" momentum states $a^\dagger(\vec{p})|0\rangle$ and $\langle 0|a(\vec{k})$. (These aren't quite external states for $\Delta V \neq 0$, but they are close.) We can generalize Wick's theorem to compute matrix elements for these "external" states. Define the contractions

$$\overbracket{a_k a_p^\dagger} = [a_k, a_p^\dagger], \qquad \overbracket{\phi_I(x) a_p^\dagger} = [\phi_I(x), a_p^\dagger], \qquad \overbracket{a_k \phi_I}(x) = [a_k, \phi_I(x)]\ .$$

   The generalization of Wick's theorem is then

$$(a_k)^\ell T\{\phi_{I_1}\dots\phi_{I_m}\}(a_p^\dagger)^n = N\left\{(a_k)^\ell\phi_1\dots\phi_m(a_p^\dagger)^n + \text{all contractions}\right\}\ .$$

   a) i) Compute $\overbracket{\phi_I(x) a_k^\dagger}$ and $\overbracket{a_k \phi_I}(x)$.

      ii) Show that this generalization of Wick's theorem is all we need to compute the matrix elements of time-ordered products of field operators between incoming and outgoing "external" momentum states.

iii) Show that if all incoming momenta are different from all outgoing momenta, the contractions between a_k and $a_p^\dagger$ can be neglected.

b) Use these results to compute the leading (in powers of g) non-zero contribution to the matrix element

$$\langle 0|a(\vec{p}_3)a(\vec{p}_4)\, T\left\{\exp[-i\int d^4z\, g\phi_I^3(z)/3!]\right\} a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle$$

Assume that $\vec{p}_1$ and $\vec{p}_2$ are different from $\vec{p}_3$ and $\vec{p}_4$, and leave your result in terms of D_F functions.

c) Evaluate the result of part b) by plugging in the explicit form of $D_F(x-y)$.

5. Complex scalars, particles and anti-particles.

Consider the quantum theory of a free complex scalar field from question # 2 in `hw-03`. This theory has two types of particle excitations, corresponding to the two creation operators $a^\dagger(\vec{p})$ and $b^\dagger(\vec{k})$. These two particle types have equal masses but opposite values of the conserved charge. For this reason, we say that one particle type is a *particle* and the other is its *anti-particle*.

a) Compute $\langle 0|\Phi(x)|\vec{p}_{(a,b)}\rangle$, $\langle\vec{p}_{(a,b)}|\Phi(x)|0\rangle$, $\langle 0|\Phi^\dagger(x)|\vec{p}_{(a,b)}\rangle$, and $\langle\vec{p}_{(a,b)}|\Phi^\dagger(x)|0\rangle$, where $|\vec{p}_a\rangle := a^\dagger(\vec{p})|0\rangle$, $|\vec{p}_b\rangle := b^\dagger(\vec{p})|0\rangle$, and so on.

b) Show that $\Phi^\dagger(x)|0\rangle$ is a linear combination of $|\vec{p}_a\rangle$ states, $\langle 0|\Phi^\dagger(x)$ is a linear combination of $\langle\vec{p}_b|$ states, and find the corresponding result for $\Phi(x)$ as well.

c) Use this to argue that charge conservation implies $\langle\Phi_1\Phi_2\rangle = \langle\Phi_1^\dagger\Phi_2^\dagger\rangle = 0$, but allows $\langle\Phi_1\Phi_2^\dagger\rangle$ and $\langle\Phi_1^\dagger\Phi_2\rangle$ to be non-zero (where $\Phi_1 = \Phi(x_1), \ldots$).

d) Generalize the generalized Wick's theorem from #4 to this theory for computing matrix elements of time-ordered field operators between external momentum states in the free theory. Just state the result, don't actually try to prove it.

e) Use your generalization to compute the following matrix elements in the free theory:

i) $\langle\vec{q}_a,\vec{k}_a|T\{\int d^4z\,|\Phi(z)|^4\}|\vec{p}_a,\vec{r}_a\rangle$

ii) $\langle\vec{q}_a,\vec{k}_b|T\{\int d^4z\,|\Phi(z)|^4\}|\vec{p}_a,\vec{r}_b\rangle$.

iii) $\langle\vec{q}_b,\vec{k}_b|T\{\int d^4z\,|\Phi(z)|^4\}|\vec{p}_a,\vec{r}_a\rangle$.

Relate your results to charge conservation. You may assume that the initial and final momenta are all different.