PHYS 526 Homework #1

Due: Sept. 17, 2013

0. Read Chs.1-3 of Srednicki. It is a nice textbook to own, and you can download a preliminary version of it here:
`http://web.physics.ucsb.edu/~mark/qft.html` .

1. Electromagnetism from a Lagrangian.

   a) Write the components of the field tensor $F_{\mu\nu}$ in terms of the components of the electric and magnetic fields. (*Hint:* $\sum_k \epsilon^{ijk}\epsilon^{\ell mk} = \delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}$.)

   b) Show that

   $$S_{em} = \int d^4x \, \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \int d^4x \, -\frac{1}{2}A^\mu \left(-\eta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu\right) A^\nu \ .$$

   Hint: integrate by parts and remember that surface terms vanish.

   c) Use this result to find the equations of motion for the A_μ fields.

   d) Show that these equations, together with the definitions of $\vec{E}$ and $\vec{B}$ in terms of potentials, are equivalent to Maxwell's equations with no sources.

2. Suppose we have a theory with two real scalar fields ϕ_1 and ϕ_2 and the action

   $$S = \int d^4x \, \mathscr{L} = \int d^4x \, \left(\frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2)\right) \ .$$

   It is convenient to rewrite these two real fields in terms of a single complex field Φ and its conjugate Φ^*:

   $$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \ , \quad \Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \ .$$

   a) Rewrite the action in terms of Φ and Φ^*.

   b) Find the equations of motion for Φ and Φ^* directly, without referring to ϕ_1 and ϕ_2.
   Hint: in this problem, treat Φ and Φ^* as independent generalized coordinates.

   c) Find the momenta conjugate to Φ and Φ^*. Use them to construct the Hamiltonian.

   d) Show that

   $$\Phi(x) = \int \widetilde{dk} \, \left[a(\vec{k})e^{-ik\cdot x} + b^*(\vec{k})e^{ik\cdot x}\right] \ ,$$

   where $k^0 = E_k = \sqrt{\vec{k}^2 + m^2}$, is a solution to the equations of motion for any complex functions $a(\vec{k})$ and $b(\vec{k})$. It turns out that any solution of the equations of motion for Φ can be written in this way (for some functions a and b).

e) Write Π, the momentum conjugate to Φ, in terms of a and b.

f) Write the Hamiltonian in terms of a and b. You should simplify the result until there is only one $\widetilde{dk}$ integral left.
Hint: remember that exponentials can integrate to delta functions.

3. A symmetry.

a) Show that the Lagrangian you found above has a symmetry under

$$\Phi \to e^{i\alpha}\Phi$$

for any constant α.

b) Find the conserved Noether current j^μ corresponding to this symmetry.
Hint: again, treat Φ and Φ^* as independent generalized variables.

c) Use the equations of motion to show that the current is conserved, $\partial_\mu j^\mu = 0$.

d) Write the corresponding conserved charge in terms of the functions a and b. Simplify the result so there is only one $\widetilde{dk}$ left at the end.

e) Suppose the transformation parameter (α) isn't a constant, but varies over space-time: $\alpha = \alpha(x)$. Is the rephasing still a symmetry of the theory?

4. Fun with exponentials.

a) A function of a matrix should be viewed as a formal power series in the matrix.

i) If $M = diag(m_1, m_2, \ldots, m_n)$ is an $n \times n$ diagonal matrix, show that $e^M = diag(e^{m_1}, e^{m_2}, \ldots, e^{m_n})$.

ii) Generalize this to show that $f(M) = diag(f(m_1), f(m_2), \ldots, f(m_n))$.

iii) Suppose that P is not diagonal, but $U^\dagger P U = diag(p_1, \ldots, p_n)$ is, where U is unitary. Show that $e^P = U\, diag(e^{p_1}, \ldots, e^{p_n})\, U^\dagger$.

b) In the Schrödinger picture of QM, states typically evolve in time by $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ with $U(t) = e^{-iHt}$ and H independent of time.

i) Prove that $U(t)$ is unitary.

ii) Show that $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, where $U(t, t_0) = U(t)U^\dagger(t_0)$.

iii) Show that $U(t_1, t_1) = 1$, $U^\dagger(t_2, t_1) = U^{-1}(t_2, t_1) = U(t_1, t_2)$, and $U(t_2, t_0)U(t_0, t_1) = U(t_2, t_1)$.

iv) Use these results to derive the Schrödinger equation: $\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle$.

c) In the Heisenberg picture of QM, states are time independent while operators evolve according to $\mathcal{O}(t) = U^\dagger(t)\mathcal{O}(0)U(t)$.

i) Show that the Schrödinger and Heisenberg pictures predict the same value for any operator expectation value in the system.

ii) Derive the Heisenberg-picture operator equation of motion: $\frac{d}{dt}\mathcal{O}(t) = i[H, \mathcal{O}(t)]$.