PHYS 526 Notes #4: Feynman Rules and Scattering

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We now have a perturbative prescription to compute the vacuum matrix elements of time-ordered products of fields. It's nice to be able to compute something, even approximately, but what we would really like is to be able to predict are physically observable quantities. In this note we will show how to relate time-ordered matrix elements to quantum amplitudes for particle scattering.

1 Asymptotic States

Our first goal is to connect n-point functions to incoming and outgoing particles travelling from and to spacetime infinity. For this, we will use two primary results: the Källén-Symanzik spectral decomposition, and the Lehmann-Symanzik-Zimmermann formula.

1.1 Spectral Decomposition

To begin, let us think about the effect of translations and Lorentz transformations on the scalar fields in the interacting theory (with some unspecified $\Delta V(\phi)$). Just like in the free theory, spacetime translations are symmetries of the theory and there are corresponding conserved 4-momentum operators $P^\mu = (H, \vec{P})$. In particular, we have

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x} \ . \tag{1}$$

Since spacetime translations commute and the operators are Hermitian, there exists a basis of P^μ eigenstates that spans the Hilbert space.

We have already made a couple of assumptions about the structure of these states. The first is that there is a vacuum state $|\Omega\rangle$ such that

$$P^\mu|\Omega\rangle = 0 \ . \tag{2}$$

The second assumption is that the next set of states up in energy are isolated one-particle states $|\vec{p}\rangle$ with

$$P^\mu|\vec{p}\rangle = p^\mu|\vec{p}\rangle \ , \tag{3}$$

with $p^0 = \sqrt{\vec{p}^2 + M^2}$ for some constant $M^2 > 0$. More precisely, we assume there is a unique one-particle state with this value of M for every possible value of the 3-momentum $\vec{p}$. We will normalize these states in the same way as the momentum states of the free theory,

$$\langle\vec{p}|\vec{k}\rangle = (2\pi)^3 2p^0\delta^{(3)}(\vec{p} - \vec{k}) \ . \tag{4}$$

For the remaining 4-momentum eigenstates $|\psi'_{\vec{p}}\rangle$ above the one-particle states, we make no further assumptions aside from a mass gap. By this, we mean that

$$P^\mu|\psi'_{\vec{p}}\rangle = p^\mu_{\psi'}|\psi'_{\vec{p}}\rangle \tag{5}$$

with $p^0_\psi = E_{\psi'}(\vec{p}) = \sqrt{\vec{p}^2 + M^2_{\psi'}}$ with $M^2_{\psi'} > M^2$. Note that since $P^2 = E^2 - \vec{p}^2$ is Lorentz-invariant, it must be equal to a constant and thus the energy must take this general form. Although we don't specify what the $|\psi'_{\vec{p}}\rangle$ states are, they include multiparticle states with total 3-momentum $\vec{p}_{\psi'}$ and possibly also single-particle bound states with mass greater than M. Collectively, we will refer to all the states above the vacuum (one-particle and higher) as $\{|\psi_{\vec{p}}\rangle\}$

It is also useful to think about the effects of Lorentz boosts on the system. These act on spacetime according to

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \ . \tag{6}$$

We will study these in more detail later on in the course, but for now, all we need to know as that such boosts can be implemented by an operator $U(\Lambda)$ on the Hilbert space. This operator does not alter the vacuum (by assumption):

$$U|\Omega\rangle = |\Omega\rangle \ . \tag{7}$$

On the field operator, we have

$$U(\Lambda)\phi(x)U^{-1}(\Lambda) = \phi(\Lambda^{-1}x) \ . \tag{8}$$

This implies $U(\Lambda)\phi(0)U^{-1}(\Lambda) = \phi(0)$. For the states carrying momentum (one-particle or otherwise), we can always rewrite them as a boost operator $U(\Lambda_{\vec{p}})$ acting on a state with zero 3-momentum:

$$|\psi_{\vec{p}}\rangle = U(\Lambda_{\vec{p}})|\psi_0\rangle \ . \tag{9}$$

In this way, we can think of all the different one-particle states as boosts of the state with a single particle at rest, $p^\mu = (M, \vec{0})$.

The completeness of the set of momentum eigenstates can be written as

$$\mathbb{I} = |\Omega\rangle\langle\Omega| + \sum_\psi \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\psi(\vec{p})}|\psi_{\vec{p}}\rangle\langle\psi_{\vec{p}}| \ . \tag{10}$$

Here, the sum over ψ includes the one-particle and other states, and it only runs over states that are not related to each other by a Lorentz transformation. In other words, it runs only over the $|\psi\rangle_0$ states since the rest can be obtained by boosting.

Let us now insert this resolution of the identity into the matrix element of a pair of fields:

$$\begin{aligned}
\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle &= \langle\Omega|\phi(0)e^{-iP\cdot(x_1-x_2)}\phi(0)|\Omega\rangle && (11)\\
&= |\langle\Omega|\phi(0)|\Omega\rangle|^2 + \sum_\psi \int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip_\psi\cdot(x_1-x_2)}}{2E_\psi(\vec{p})}|\langle\Omega|\phi(0)|\psi_{\vec{p}}\rangle|^2 && (12)\\
&= 0 + \sum_\psi \int \frac{d^3p}{(2\pi)^3)}\frac{e^{-ip_\psi\cdot(x_1-x_2)}}{2E_\psi(\vec{p})}|\langle\Omega|\phi(0)|\psi_0\rangle|^2 && (13)
\end{aligned}$$

In the last line we have used $|\psi_{\vec{p}}\rangle = U(\Lambda_{\vec{p}})|\psi_0\rangle$ and we have taken $\langle\Omega|\phi(0)|\Omega\rangle = 0$. There is nothing to guarantee that this quantity vanishes, but if it does not, we can use a shifted field variable $\phi'(x) = \phi(x) - |\langle\Omega|\phi(0)|\Omega\rangle|$ for which this is true.

Using the fact that $E_\psi(\vec{p}) = \sqrt{\vec{p}^2 + M_\psi^2}$ and putting in a time ordering, Eq. (13) implies

$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\}|\Omega\rangle &= \sum_\psi \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - M_\psi^2 + i\epsilon}e^{-ip\cdot(x_1-x_2)}|\langle\Omega|\phi(0)|\psi_0\rangle|^2 \qquad (14)\\ &= \int_0^\infty \frac{ds}{2\pi}\,\rho(s)\,D_F(x_1-x_2;s)\ , \qquad (15)\end{aligned}$$

where $D_F(x_1 - x_2; s)$ is the Feynman propagator for a free field of mass $m^2 = s$, and $\rho(s)$ is defined to be

$$\rho(s) = \sum_\psi (2\pi)\delta(s - M_\psi^2)|\langle\Omega|\phi(0)|\psi_0\rangle|^2 \qquad (16)$$

The result of Eq. (15) is called the Källén-Lehmann spectral representation. In the free theory, we would just have $\rho(s) = 2\pi\delta(s - m^2)$ and the 2-point function reduces to the Feynman propagator with $s = m^2$. In the interacting theory, $\rho(s)$ is a non-trivial *spectral function* that characterizes the set of excitations in the theory. With our assumption of an isolated one-particle state of mass M, it takes the form

$$\rho(s) = 2\pi\,Z\,\delta(s - M^2) + \bar{\rho}(s)\ , \qquad (17)$$

where $Z = |\langle\Omega|\phi(0)|\vec{p}\rangle|^2 > 0$. We illustrate the form of $\rho(s)$ under this assumption in Fig. 1. Above the one-particle state, there could be isolated bound states, and there will definitely be a continuum of multi-particle states at $s \geq (2M)^2$. As a function in the complex s plane, distinct particles correspond to isolated poles, while the continuum of multiple particles corresponds to a branch cut along the real line.

1.2 Lehmann, Symanzik, and Zimmermann (LSZ)

The LSZ reduction formula is one of the key tools in perturbative quantum field theory. It relates the vacuum matrix elements of time-ordered products of field operators to the matrix elements for particle scattering. We will not go through the proof of the formula (which can be found in Peskin & Schroeder [1]), but we will try to motivate it.

Consider first our spectral representation of the 2-point function. Let us Fourier transform the result to momentum space:

$$\int d^4x\; e^{ik\cdot x}\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle = \int_0^\infty \frac{ds}{2\pi}\,\rho(s)\frac{i}{k^2 - s + i\epsilon}\ . \qquad (18)$$

If we now apply our assumption of an isolated one-particle state, this becomes

$$\int d^4x\; e^{ik\cdot x}\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle = \frac{i\,Z}{k^2 - M^2 + i\epsilon} + \int_{>M^2}^\infty \frac{ds}{2\pi}\,\bar{\rho}(s)\,\frac{i}{k^2 - s + i\epsilon}\ . \qquad (19)$$

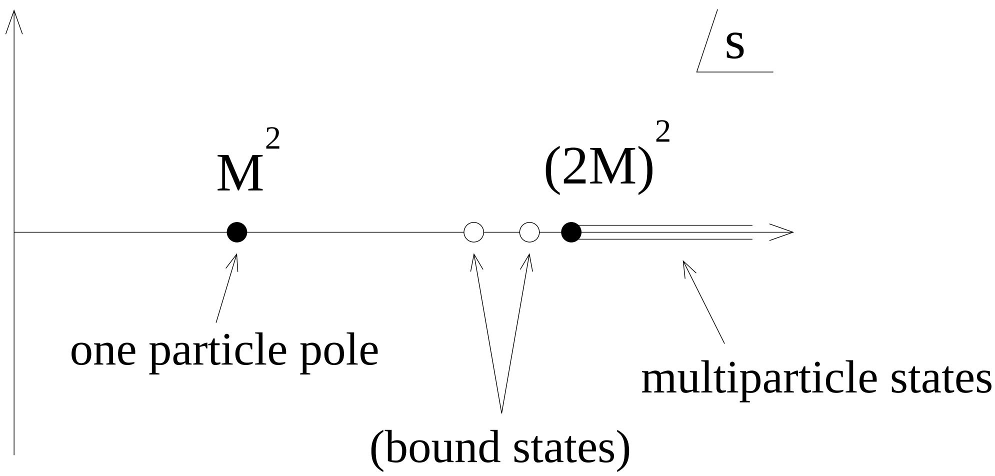

Figure 1: Schematic depiction of the spectral function in a theory with a weak interaction.

Viewed as a function in the complex k^0 plane, this quantity has an isolated pole at $k^0 = +\sqrt{\vec{k}^2 + M^2 + i\epsilon}$, corresponding to the physical energy of a one-particle state with momentum $\vec{k}$ and mass M, as well as a more complicated and undertermined structure further out in the k^0 plane. This means that if we want to focus on the one-particle state and ignore the rest of the junk that comes up, we just need to isolate the k^0 pole.

The LSZ reduction formula is just a generalization of this observation. It states that the *connected amplitude* for an initial state with m well-separated particles in the initial state at $t \to -\infty$ with 3-momenta $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_m$ to go to a final state with n well-separated particles at $t \to +\infty$ with 3-momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ is given by

$$\begin{aligned} \langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_1 \ldots \vec{k}_m \rangle_c \;=\; & \\ & \left(\lim_{k_1^2 \to M^2} \ldots \lim_{k_m^2 \to M^2} \right) \left(\lim_{p_1^2 \to M^2} \ldots \lim_{p_n^2 \to M^2} \right) \frac{i^{n+m}}{(\sqrt{Z})^{(m+n)}} \\ & \int d^4 z_1 \, e^{-ik_1 \cdot z_1} (\partial_{z_1}^2 + M^2) \ldots \int d^4 z_m \, e^{-ik_m \cdot z_m} (\partial_{z_m}^2 + M^2) \\ & \int d^4 x_1 \, e^{ip_1 \cdot x_1} (\partial_{x_1}^2 + M^2) \ldots \int d^4 x_n \, e^{ip_n \cdot x_n} (\partial_{x_n}^2 + M^2) \\ & \langle \Omega | T\{\phi(x_1) \ldots \phi(x_n) \phi(z_1) \ldots \phi(z_m)\} | \Omega \rangle \end{aligned} \tag{20}$$

$$\begin{aligned} = & \left(\lim_{k_1^2 \to M^2} \ldots \lim_{k_m^2 \to M^2} \right) \left(\lim_{p_1^2 \to M^2} \ldots \lim_{p_n^2 \to M^2} \right) \frac{i^{n+m}}{(\sqrt{Z})^{(m+n)}} \\ & \prod_{i=1}^{m} \left[\int d^4 z_i \, e^{-ik_i \cdot z_i} (-k_i^2 + M^2) \right] \prod_{j=1}^{n} \left[\int d^4 x_j \, e^{ip_j \cdot x_j} (-p_j^2 + M^2) \right] \\ & \langle \Omega | T\{\phi(x_1) \ldots \phi(x_m) \phi(z_1) \ldots \phi(z_n)\} | \Omega \rangle \end{aligned} \tag{21}$$

To get the second line, we have integrated by parts in each of the integrals.

As an operational tool, the LSZ formula can be applied to find the matrix element for $\langle\vec{p}_1\ldots\vec{p}_n|\vec{k}_1\ldots\vec{k}_m\rangle_c$ by following a few simple steps. First, compute the $(m+n)$-point function and take its Fourier transform using $\int d^4z\, e^{-ik\cdot z}$ for all incoming particles and $\int d^4x\, e^{ip\cdot x}$ for all outgoing ones. Next, identify all the poles at $k_i^2 = M^2$ and $p_j^2 = M^2$. Cancel off the poles in the terms that contain a product of all of them and set everything else to zero. Finally, take the limit $k_i^2 \to M^2$ and $p_j^2 \to M^2$. This procedure isolates the part of the $(m+n)$-point function that corresponds to m isolated initial particles and n isolated final particles. The portion of the $(m+n)$-point function that does not have all the necessary poles vanishes when it is multiplied by the $(p^2 - M^2)$ factors and the $p^2 \to M^2$ limit is taken.[1]

Let us apply this result to the 3-point function in the interacting theory with $\Delta V = g\phi^3/3!$ that we computed to order g^1 in note-3 (see Fig.1). Recall that we had

$$\begin{aligned} G^{(3)}(x_1,x_2,x_3) &= (-ig)\int d^4z\, D_F(x_1-z)D_F(x_2-z)D_F(x_3-z) \\ &\quad +\frac{(-ig)}{2}\int d^4z\, D_F(z-z)D_F(x_3-z)D_F(x_1-x_2) + (\text{permutations})\ . \end{aligned} \tag{22}$$

Fourier transforming the first term (with three initial states) produces

$$T_1 = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\,(-ig)\,\frac{i}{p_1^2-M^2}\frac{i}{p_2^2-M^2}\frac{i}{p_3^2-M^2}\ . \tag{23}$$

This has three poles right where we expect them. On the other hand, the Fourier transform of the second term is

$$T_2 = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\left[\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M^2}\right]\frac{i}{p_1^2-M^2}\frac{i}{p_3^2-M^2}\,(2\pi)^4\delta^{(4)}(p_3)\ , \tag{24}$$

which disappears when it is multiplied by $(p_1^2-M^2)(p_2^2-M^2)(p_3^2-M^2)$, as do the permutations. Applying the LSZ formula therefore produces

$$\langle\Omega|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_c = (-ig)\times(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\ . \tag{25}$$

Note that we would have obtained the nearly same result if we had put some of the particles in the final state. The only difference would have been $p_i \to -p_i$ in the delta function for every final-state particle. The delta function therefore enforces the overall conservation of energy and momentum. It turns out that this is a universal feature.

The terms that vanish when the LSZ formula is applied to the 3-point function (at order g^1) correspond to *disconnected diagrams*. These diagrams contain one or more particles that just pass through without interacting with the others at all. They are not what we are interested in when we compute scattering, and it is a useful feature of the LSZ formula that they are automatically removed. This is also why we said that the resulting matrix element is connected.

[1] I'm being sloppy with the $\imath\epsilon$ factors here, but feel free to put them in as needed.

1.3 The S-Matrix

Our result for matrix elements between incoming and outgoing particles that are well-separated at spacetime infinity is frequently written in terms of an S-matrix. If we restrict ourselves to well-separated sets of particles at $t \to \pm\infty$, we can map these states in an obvious way to the set of momentum eigenstates of a free field theory. In this picture, the free states at $t \to -\infty$ are called IN states and those at $t \to +\infty$ are called OUT states.

The matrix elements of interest for scattering are therefore

$$ {}_{OUT}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_1 \dots \vec{k}_m \rangle_{IN} \ , \tag{26} $$

where the value of the inner product is defined to be equal to the value obtained between the corresponding states in the interacting theory. The S matrix is defined to be the mapping between these two different but equivalent Hilbert spaces:

$$ |\vec{k}_1 \dots \vec{k}_m\rangle_{IN} = S|\vec{k}_1 \dots \vec{k}_m\rangle_{OUT} \ . \tag{27} $$

In other words, the S-matrix maps each OUT state to the corresponding element of the IN Hilbert space, as determined by the dynamics of the interacting theory. It is a map from one Hilbert space defined at $t \to -\infty$ to an equivalent one at $t \to +\infty$, and it it is invertible. In the absence of scattering, the S matrix is unity, which is what we found for the free theory.

The key property of the S matrix is that it is unitary. Note that we have

$$ \begin{aligned} \delta(\{\vec{k}_i\} - \{\vec{p}_i\}) &= {}_{IN}\langle \{\vec{k}_i\} | \{\vec{p}_i\} \rangle_{IN} && (28) \\ &= {}_{OUT}\langle \{\vec{k}_i\} | S^\dagger S | \{\vec{p}_i\} \rangle_{OUT} \ . && (29) \end{aligned} $$

The orthogonality of the OUT states implies that $S^\dagger S = \mathbb{I}$. Physically, the unitarity of the S matrix corresponds to the conservation of probability, in the sense that the sum over all the IN to OUT squared matrix elements adds up to one. This is also expected from our assumption that the time evolution in quantum mechanics is unitary.

We will frequently write

$$ S = e^{iT} \simeq \mathbb{I} + iT \ , \tag{30} $$

where the T matrix is Hermitian. The unit term in the expansion of the S matrix corresponds to the case of no scattering. We will therefore be interested primarily in the T matrix.

2 Feynman Rules in Momentum Space

In the LSZ formula, all the position variables of the n-point function are Fourier transformed. For this reason, it is useful to formulate Feynman rules directly in momentum space, where the position variables do not appear at all. Once we have the transformed n-point function, it is trivial to apply the LSZ formula.

2.1 Feynman Rules for the Transformed n-Point Function

We will continue to study the real scalar theory with $\Delta V = g\phi^3/3!$. Let us define the Fourier transform of the n-point function by

$$(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{n} p_i\right)\tilde{G}^{(n)}(p_1,\ldots,p_n) = \left(\prod_{i=1}^{n}\int d^4x_i\, e^{-ip_i\cdot x_i}\right)\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle\ . \quad (31)$$

The Feynman rules for the order-g^M portion of the momentum space n-point function are:

1. Draw an *external line* for each momentum p_i with one fixed end and one free end.
2. Put in M vertices, each with three lines (with free ends) coming out of it.
3. Assemble all the Feynman diagrams by connecting the free ends of the external lines and the vertex lines to each other in pairs in all possible ways.
4. Remove all the diagrams with vacuum bubbles and any diagram that has one lines with an unconnected free end.
5. Assign a value to each diagram:

   a) Each line gets a propagator factor of $i/(p^2-M^2+i\epsilon)$. The four-momentum p is equal to p_i on external lines. The momentum on any other *internal* line is undertermined at this point, so call it whatever you like, q_j say.

   b) Write a factor of $-ig$ for each vertex.

   c) Add a factor of $(2\pi)^4\delta^{(4)}(\sum k_i)$ for each vertex, where the sum runs over all momenta (internal or external) flowing into the vertex (with $k_i \to -k_i$ if the momentum is flowing out of the vertex). Also, if a pair of external lines p_i and p_j are connected, add a factor of $(2\pi)^4\delta^{(4)}(p_i+p_j)$. The delta functions arise automatically from the Fourier transforms, and they have the effect of imposing the conservation of four-momentum at each vertex.

   d) Integrate over each of the internal momenta: $\int d^4q_j/(2\pi)^4$.

   e) Multiply each diagram by its symmetry factor.

The resulting sum of the diagrams is the order-g^M contribution to $\tilde{G}^{(n)}(p_1,\ldots,p_n)$ times the overall delta function.

All this may sound complicated, but it is really pretty easy once you see a few examples and get the hang of it. Consider first the 2-point function at leading order:

$$(2\pi)^4\delta^{(4)}(p_1+p_2)\,\tilde{G}^{(2)}(p_1,p_2) = (2\pi)^4\delta^{(4)}(p_1+p_2)\,\frac{i}{p_1^2-M^2} \quad (32)$$

$$\Rightarrow\quad \tilde{G}^{(2)}(p) = \frac{i}{p^2-M^2}\ . \quad (33)$$

Note that this would normally depend on two arguments, but since $p_1+p_2=0$ is enforced by the overall delta function, it is conventional to write it with just a single argument

$p = p_1 = -p_2$. Going to order g^2, the relevant diagrams are just like in Fig.2 of notes-3. After cancelling off the overall factor of $(2\pi)^4\delta^{(4)}(p_1+p_2)$ we find that the first diagram (from left to right) is equal to

$$D_1 = \left[\frac{1}{2}\right]\left(\frac{i}{p^2 - M^2}\right)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - M^2}\frac{i}{(p-q)^2 - M^2}\ , \tag{34}$$

the second is

$$D_2 = \left[\frac{1}{2}\right]\left(\frac{i}{p^2 - M^2}\right)^2 \frac{i}{(0 - M^2)}\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - M^2}\ , \tag{35}$$

while the third is given by

$$D_3 = \left[\frac{1}{4}\right]\left(\frac{i}{p^2 - M^2}\right)^2 \left[\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - M^2}\right]^2 (2\pi)^4\delta^{(4)}(p)\ . \tag{36}$$

Of the three, this last piece is the only disconnected one. In each case, the q integrations run over the momenta that are not completely fixed by momentum conservation.

Applying our Feynman rules to the 3-point function at order-g^1, it is straightforward to check that the results of Eqs. (23,24) are reproduced.

2.2 Feynman Rules for Connected Amplitudes

With Feynman rules formulated in momentum space, it is now really easy to compute the connected amplitudes for scattering using the LSZ formula. The procedure to find the amplitude for $\langle \vec{p}_1\vec{p}_2\ldots\vec{p}_n|\vec{k}_1\ldots\vec{k}_m\rangle_c$ at order g^M is:

1. Use the rules outlined above to find the perturbative value of $\tilde{G}^{(m+n)}(-p_1, -p_2, \ldots, -p_n, k_1, k_2)$ keeping terms up to order g^M.

2. Remove all the diagrams that are not completely connected.

3. Cancel off all the external propagator factors. Equivalently, multiply each diagram by $(-i)^{n+m}\prod_{i=1}^n(p_i^2 - M^2 + i\epsilon)\ \times \prod_{j=1}^m(k_j^2 - M^2 + i\epsilon)$.

4. Find $\sqrt{Z}$ perturbatively by computing the 2-point function up to order g^M, multiply everything by $1/(\sqrt{Z})^{(n+m)}$, and keep terms only up to order g^M.

5. Take the limit $p_i^2 \to M^2$ for all external momenta.

The final result of all these steps is $\langle \vec{p}_1\vec{p}_2\ldots\vec{p}_n|\vec{k}_1\ldots\vec{k}_m\rangle_c$

As a simple example, let us compute the connected amplitude for $\langle\vec{p}_2\vec{p}_3|\vec{p}_1\rangle$ at order g^1. The only connected part is the first term T_1 found in Eq. (23). Following our rules, the amplitude at this order is

$$\langle\vec{p}_2\vec{p}_3|\vec{p}_1\rangle = (-ig)\times(2\pi)^4\delta^{(4)}(p_2 + p_3 - p_1)\ . \tag{37}$$

This result explains why we assign a value of $-ig$ to each vertex.

The only slightly tricky part about these rules are the factors of $\sqrt{Z}$. Recall that this factored emerged in the Källén-Lehmann spectral representation in the form of Eqs. (17,19). They are called wavefunction renormalization factors, and they come from interactions changing the normalization of the field as defined by the residue of the one-particle pole of the 2-point function. In the present case, we have $Z = 1 + Ag^2 + \ldots$, so we can take $Z = 1$ if we are working to leading non-trivial order in the coupling. This will be the case for most of what we will do, but we will come back later on in the course to discuss how to deal with the wavefunction factors properly.

3 Scattering and Decays

We are now equipped to begin computing physical observables in the form of scattering cross sections and particle decay rates. These are the main types of observables that one uses perturbative quantum field theory to calculate.

3.1 Scattering Cross Sections

A typical scattering experiment consists of an initial state of two well-separated particles that collide with each other to create a final state with n independent particles that propagate off to infinity (or thereabouts). Using perturbation theory, we can compute the vacuum matrix element $\langle\Omega|T\{\phi(x_3)\ldots\phi(x_{n+2})\phi(z_1)\phi(z_2)|\Omega\rangle$. The LSZ formula then allows us to relate this vacuum matrix element to the connected matrix element $\langle\vec{p}_2\ldots\vec{p}_{n+2}|\vec{k}_1\vec{k}_2\rangle_c$. This quantity is related to the scattering amplitude $\mathcal{M}$ by

$$-i\mathcal{M} = \langle\vec{p}_3\ldots\vec{p}_{n+2}|\vec{k}_1\vec{k}_2\rangle_c \Big/ (2\pi)^4\delta^{(4)}(k_1+k_2-p_3-\ldots-p_{n+2}) \ . \tag{38}$$

Since the connected matrix element is always proportional to the overall four-momentum delta function, this equation just says that we should cancel off this delta function to get the amplitude.

The $2 \to n$ scattering cross section, corresponding to the total probability of non-trivial scattering per unit initial flux, is related to the scattering amplitude by

$$\sigma = \frac{S}{|\vec{v}_1-\vec{v}_2|}\frac{1}{2E_1\,2E_2}\int\frac{d^3p_3}{(2\pi)^3 2E_3}\cdots\int\frac{d^3p_{n+2}}{(2\pi)^3 2E_{n+2}}\,(2\pi)^4\delta^{(4)}(k_1+k_2-\sum_{i=3}^{n+2}p_i)\,|\mathcal{M}|^2 \ , \tag{39}$$

where $|\vec{v}_1-\vec{v}_2|$ is the magnitude of the initial relative velocity (which goes to unity in the extreme relativistic limit), and S is a combinatoric factor equal to one times $1/k!$ for every set of k identical particles in the final state. Derivations of this result can be found in Peskin & Schroeder [1], Srednicki [2], and Griffiths [3].

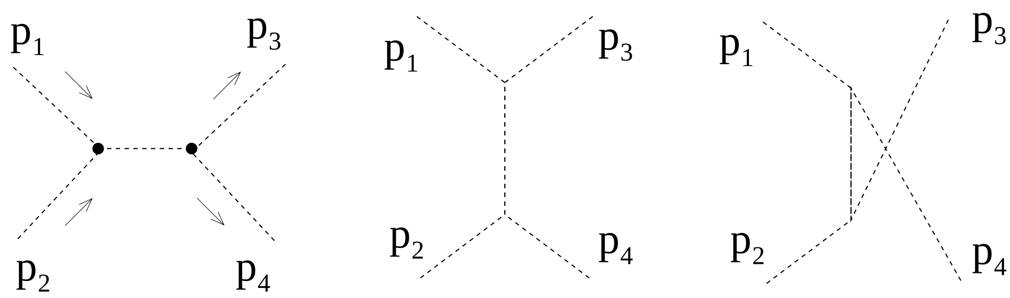

Figure 2: Feynman diagrams for *e.g. 1.*

The result of Eq. (39) has a lot going on within it, but its physical content is very simple. First, $\mathcal{M}|^2$ is the probability density for a single initial state $\vec{k}_1 + \vec{k}_2$ to scatter into the specific final state $\vec{p}_3 + \ldots + \vec{p}_{n+2}$. The delta function enforces overall four-momentum conservation. The scattering probability density is then summed over all distinct final states with a relativistic normalization. Collectively, this set of final states is often called the *phase space*. The prefactor before the integrations is a normalization to convert the result for a single initial state to the scattering probability rate per unit incident flux (= number of incident particles per unit area per unit time). At the end of the day, the cross section has units of area. The factor of S accounts for sets of indistinguishable particles.

e.g. 1. **$2 \to 2$ Scattering in the $g\phi^3/3!$ Theory**
To compute the scattering amplitude, we need to draw all the Feynman diagrams for this theory that can contribute. For now, we will only compute the leading non-trivial contribution in g. This comes from the diagrams in Fig. 2, which I have drawn with time going from left to right, and p_1 and p_2 as the initial momenta and p_3 and p_4 as the final momenta. The amplitude at this order is

$$-i\mathcal{M} = (-ig)^2 \left[\frac{i}{(p_1+p_2)^2 - M^2} + \frac{i}{(p_1-p_3)^2 - M^2} + \frac{i}{(p_1-p_4)^2 - M^2} \right] . \tag{40}$$

These three diagrams are called s-, t-, and u-channel respectively. This designation corresponds to the three Lorentz invariant combinations of momenta relevant for $2 \to 2$ scattering:

$$s = (p_1+p_2)^2 = (p_3+p_4)^2 \tag{41}$$

$$t = (p_1-p_3)^2 = (p_2-p_4)^2 \tag{42}$$

$$u = (p_1-p_4)^2 = (p_2-p_3)^2 . \tag{43}$$

Note that $s + t + u = 4M^2$. In terms of these combinations, we have

$$\mathcal{M} = g^2 \left[\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u-M^2} \right] . \tag{44}$$

e.g. 2. **Elastic Scattering in the CM Frame**
Consider the $2 \to 2$ elastic scattering of a pair of particles of mass M. We will work in the center-of-mass (CM) frame where the sum of the initial three-momenta vanishes. It is conventional to choose the z-axis in the direction of the incoming particles. Specifically,

$$p_1 = (E, 0, 0, p), \qquad p_2 = (E, 0, 0, -p) \ . \tag{45}$$

with $E = \sqrt{p^2 + M^2}$. Applying the overall conservation of energy and momentum, the outgoing momenta must take the form

$$p_3 = (E, p\sin\theta, 0, p\cos\theta), \qquad p_4 = (E, -p\sin\theta, 0, -p\cos\theta) \ , \tag{46}$$

where we have chosen to align the $\vec{p}_3$ axis to simplify the form of p_3. In terms of these values, we can express the momentum invariants s, t, and u as

$$s = 4(M^2 + p^2), \quad t = -p^2(1-\cos\theta)^2, \quad u = -p^2(1+\cos\theta)^2 \ . \tag{47}$$

The scattering cross section in this frame is

$$\begin{aligned}
\sigma &= \frac{(1/2!)}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{4E^2}\int \frac{d^3p_3}{2E_3(2\pi)^3}\int\frac{d^3p_4}{2E_4(2\pi)^3}\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,|\mathcal{M}|^2 && (48)\\
&= \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{32(2\pi)^2E^2}\int_0^\infty dp'\,p'^2\int d\Omega\,|\mathcal{M}|^2\,\delta(2E - 2\sqrt{p'^2+M^2}) && (49)\\
&= \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{64\pi E^2}\,\frac{p}{E}\left(\frac{1}{4\pi}\int d\Omega\,|\mathcal{M}|^2\right) \ . && (50)
\end{aligned}$$

Sometimes $2 \to 2$ scattering is characterized by the differential cross section per unit solid angle, given in this case by

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{256\pi^2E^2}\,\frac{p}{E}\,|\mathcal{M}|^2 \ , \tag{51}$$

where $d\Omega = d(\cos\theta)\,d\phi$.

3.2 Decays

The decays of an unstable particle are probabilistic, but are characterized by an average decay rate Γ. Specifically, given an initial sample of N_0 particles at time $t = 0$, the number of particles after time t is

$$N(t) = N_0 e^{-\Gamma t} \ . \tag{52}$$

The lifetime τ of a particle species is defined to be

$$\tau = 1/\Gamma \ . \tag{53}$$

Sometimes you will also hear of half-lives, given by $\tau_{1/2} = \tau \ \ln 2$. In natural units, the decay rate has units of mass.

The decay rate can be computed using the LSZ formula, even though this formula only really applies to stable particles that are able to propagate off to infinity. However, it turns out that the LSZ formula is also a good approximation for particles that are unstable but whose decay rates are very slow relative to their mass. In this case, the partial rate for an unstable particle of mass M at rest to decay to a final state containing n particles $(1 \to 2 + 3 + \ldots + n + 1)$ is

$$\Gamma(1 \to n) = \frac{S}{2M} \int \frac{d^3p_2}{2E_2(2\pi)^3} \ldots \int \frac{d^3p_2}{2E_2(2\pi)^3} \, (2\pi)^4 \delta^{(4)}(p_1 - \sum_{i=1}^{n+1} p_i) |\mathcal{M}|^2 \ , \tag{54}$$

where $|\mathcal{M}|^2$ is the corresponding $1 \to n$ amplitude defined in the same way as for scattering, and S is the symmetry factor. The total decay rate is the sum of the partial rates Γ_f of all the individual decay channels,

$$\Gamma = \sum_f \Gamma_f = \Gamma \ \sum_f BR_f \ , \tag{55}$$

where $BR_f = \Gamma_f/\Gamma$ is the *branching ratio* to the final state f.

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