P528 Notes #9: QCD at Low Energies

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Quantum Chromodynamics (QCD) is the accepted theory of the strong force. It is an $SU(3)$ gauge theory with a *gluon* vector boson and a set of massive fermion *quark* fields transforming under the fundamental **3** irrep of the gauge group. While the underlying QCD theory is very simple, the resulting dynamics are anything but. We never actually observe quarks or gluons as free asymptotic particles. Instead, at low energies (or long distances) we only ever see colour-neutral bound states of quarks and gluons. This stands in stark contrast to QED, where we certainly do see free particles charged under the gauge group – electrons for example. The absence of free colour-charged objects is called *confinement*.

Confinement is still not completely understood at the quantitative level due to the breakdown of perturbation theory in QCD at low energies. Despite these challenges, it is still possible to construct a useful low-energy EFT for the bound states resulting from QCD confinement. Collectively these bound states are called *hadrons*, and the most important examples are *mesons* and *baryons*. The quantum numbers of these states can be matched to the colour-neutral quark operators

$$M \sim \bar{q}^i q'_j \delta^i_j \ , \qquad B \sim q_i q'_j q''_k \epsilon^{ijk} \ , \tag{1}$$

where i and j are colour indices. We discuss some of these ideas here.

1 Aspects of QCD

The fundamental QCD Lagrangian is [1, 2]

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_I \bar{q}_I (i\gamma^\mu D_\mu - m_I) q_I \ , \tag{2}$$

where $I = u,\, d,\, s,\, c,\, b,\, t$, and

$$\begin{aligned} D_\mu &= \partial_\mu + i g_s t^a_3 G^a_\mu, & (3)\\ G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f^{abc} G^b_\mu G^c_\nu \ . & (4) \end{aligned}$$

The matrices t^a_3 are the (eight) generators of the fundamental **3** irrep of $SU(3)$. Gauge charges are called *colour* while the different species of 4-component Dirac fermion quarks are called *flavours*. The masses m_I of the different quark flavours are approximately [3]

$$\begin{array}{lll} m_u \simeq 2.5 \text{ MeV} & m_d \simeq 5.3 \text{ MeV} & m_s \simeq 110 \text{ MeV} \\ m_c \simeq 1.25 \text{ GeV} & m_b \simeq 4.5 \text{ GeV} & m_t \simeq 173 \text{ GeV} \ . \end{array} \tag{5}$$

Of course, this structure fits in nicely with the rest of the SM, with the quark masses arising from electroweak symmetry breaking.

1.1 Running Couplings

A key feature of QCD (and other gauge theories) is the scale dependence of the renormalized coupling $g_s(\mu)$. Recall that μ is an arbitrary renormalization scale that we can choose as we wish. When $\mu \sim p$, the value of this coupling coincides reasonably well with the physical QCD coupling strength in a process occurring at the characteristic momentum scale p. In a generic gauge theory, the running coupling $g(\mu)$ can be obtained by measuring the coupling at one momentum scale and solving the renormalization group (RG) equation to extrapolate it to other momentum scales. At one-loop order, the RG equation is [1]

$$\frac{dg}{dt} := \beta(t) = -\frac{b}{(4\pi)^3} g^3 \ , \tag{6}$$

where the coefficient b is given by

$$b = \frac{11}{3} C_2(A) - \sum_r \frac{2}{3} T_2(r) - \sum_{r'} \frac{1}{3} T_2(r') \ , \tag{7}$$

where $C_2(A)$ is the Casimir of the adjoint ($C_2(A) = N$ for $SU(N)$), $T_2(r)$ is the trace invariant of the representation r ($T_2(N) = 1/2$ for the fundamental N irrep of $SU(N)$), $t = \ln(\mu/\mu_0)$, the first sum runs over all light 2-component fermion reps in the theory, and the second sum runs over all light. complex scalar reps. In this context, "light" implies reps with mass $m < \mu$.

Let us apply this result to QCD at high energies, $\mu \gg m_t$, so that all quarks flavours count as light. The coefficient b is then

$$b_{QCD} = \frac{11}{3} \times 3 \quad - \quad \frac{2}{3} \times \frac{1}{2} \times 2 \times 6 = 7 \ . \tag{8}$$

Here, we have used the facts that $C_2(A) = 3$ for $SU(3)$, $T_2(\mathbf{3}) = 1/2$, each quark flavour contains two 2-component fermions, and there are six flavours in total. Solving Eq. (6), we find

$$\alpha_s(\mu) \ = \ \frac{g_s^2(\mu)}{4\pi} \ = \ \frac{\alpha_s(\mu_0)}{1 + \frac{b_{QCD}}{2\pi}\alpha_s(\mu_0) \ln\left(\frac{\mu}{\mu_0}\right)} \ , \tag{9}$$

where μ_0 is some reference scale. Since the coefficient $b_{QCD} = 7$ is positive, this result (or Eq. (6)) implies that the QCD coupling strength $\alpha_2(\mu)$ becomes *weaker* going to higher energies This property is called *asymptotic freedom*, and is a very special feature of non-Abelian gauge theories.

The flip side of asymptotic freedom is that the QCD coupling strength grows stronger at lower energies. Starting from $\mu > m_t$, let us now extrapolate $\alpha_s(\mu)$ down to lower scales. Recall that as μ falls below the mass M of a heavy quark, it stops contributing to the RG running and the b coefficient in Eq. (7) changes. To match the running above and below such a mass threshold at $\mu = M$, the (leading order) condition is

$$\lim_{\mu \to M_-} g(\mu) = \lim_{\mu \to M_+} g(\mu) \ . \tag{10}$$

That is, the running coupling is continuous across the mass threshold. Applying this to QCD, the coefficients are

$$b_{QCD} = \begin{cases} 7 & \mu > m_t \\ 23/3 & m_b < \mu < m_t \\ 25/3 & m_c < \mu < m_b \\ \dots & \end{cases} . \tag{11}$$

To solve for the numerical value of $\alpha_s(\mu)$, we also need a reference input value taken from experiment. The most common convention is to use $\mu = m_Z$, where it is found that [3]

$$\alpha_s(m_Z) \;=\; 0.118 \pm 0.002 \; . \tag{12}$$

Together with Eq. (6), we can now solve for $\alpha_s(\mu)$ at any scale μ.

This treatment of $\alpha_s(\mu)$ is based on perturbation theory, and it breaks down as the coupling becomes large, approaching 4π. Numerically, this occurs at a scale that is very close to where the denominator in Eq. (9) vanishes. Let us define this scale to be $\mu = \Lambda_{QCD}$. In terms of it, we can rewrite Eq. (9) as

$$\alpha_s(\mu) = \frac{2\pi}{b_{QCD}\ln(\mu/\Lambda_{QCD})} \; . \tag{13}$$

The appearance of a dimensionful scale from a dimensionless (but scale-dependent) coupling is called *dimensional transmutation*. Numerically, $\Lambda_{QCD} \simeq 200$ MeV, and this value characterizes the onset of strong coupling in QCD. In practice, QCD becomes strongly-coupled a little earlier than this, near $E \sim 1$ GeV, which is roughly the mass scale of the light baryons.

1.2 Confinement

Asymptotic freedom suggests a qualitative picture of confinement. Quarks and gluons are weakly-coupled at high energy, but bind very strongly at low energy as the QCD coupling grows large. The lowest energy states correspond to configurations in which all the QCD flux lines connect on themselves, corresponding to colour-neutral bound states.

Another way to think about this is that low energies correspond to large distances, and we expect that a quark-antiquark pair will bind more and more strongly if we were to try to pull them apart. Indeed, it is found that the $q\bar{q}$ potential energy at separation r is modelled reasonably well by [4]

$$V(r) \sim -\frac{\alpha_s(r^{-1})}{r} + \Lambda_{QCD}^2 r. \tag{14}$$

The first term is a familiar Coulombic attraction, while the second diverges as $r \to \infty$ and signals confinement. For $r \gtrsim \Lambda_{QCD}$, the energy density between the $q\bar{q}$ pair becomes large enough that it is energetically favourable to nucleate a $q'\bar{q}'$ pair from the vacuum to form a pair of colour-neutral mesons.

2 QCD at Low Energies

Confinement implies that the QCD degrees of freedom we observe at energies below $E \lesssim \Lambda_{QCD}$ are not quarks or gluons, but rather colour neutral objects like mesons and baryons. To describe these objects efficiently, we would like a field theory in which they appear as the dynamical fields. In other words, we want a low-energy EFT for QCD.

Such an EFT is not easy to derive. Quarks and gluons are weakly-coupled at energies well-above Λ_{QCD}, and baryons and mesons are observed to be weakly-coupled at energies well below it, but in between there is strong coupling. Therefore developing an EFT for low-energy QCD requires going beyond perturbation theory, and addressing the full dynamics of the theory. A successful but very challenging approach is to simulate the underlying gauge theory numerically on a spacetime lattice [5, 6]. A second complimentary approach, and the one we will discuss here, is to simply write an effective low-energy theory with the appropriate set of degrees of freedom and all possible interactions consistent with the underlying symmetries [7, 8, 9]. The coefficients of these interactions can be set by comparing to observation, or by computing them from QCD using lattice simulations.

2.1 Global Symmetries

Symmetries can provide enormous guidance in formulating EFTs, and this is particularly true for low-energy QCD. For QCD, the key observation is that the u and d quarks are both very light relative to Λ_{QCD}, the s quark is somewhat light, and the other quarks are relatively heavy. Thus, to study the lightest QCD degrees of freedom we should be able to integrate out the c, b, and t quarks and work only with the u, d, and s quarks.

To simplify the analysis, let us begin by considering a simplified version of QCD with only u and d quarks and no masses [7]. In this form, it is convenient to assemble these two quark flavours into left- and right-handed doublets in flavour space,

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \qquad q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} . \tag{15}$$

In terms of these doublets, the simplified two-flavour QCD Lagrangian becomes

$$\mathscr{L}_2 = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \bar{q}_L i\gamma^\mu D_\mu q_L + \bar{q}_R i\gamma^\mu D_\mu q_R , \tag{16}$$

with $D_\mu = \partial_\mu + i g_s t^a_3 G^a_\mu$ for both L and R terms.

The form of the Lagrangian in Eq. (16) has an explicit $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$ global flavour symmetry under which the fields transform as

$$q_{L\,I} \to e^{i\alpha_V} e^{i\alpha_A} L_{IJ} q_{L\,J} , \qquad q_{R\,I} \to e^{i\alpha_V} e^{-i\alpha_A} R_{IJ} q_{R\,J} , \tag{17}$$

where we have written the flavour components I, $J = u, d$ explicitly, and L and R are $SU(2)_{L,R}$ transformations for the fundamental reps in flavour space. The corresponding

Noether currents for the subgroup factors are

$$\begin{aligned} j_V^\mu &= \bar{q}\gamma^\mu q \ , & j_A^\mu &= \bar{q}\gamma^\mu\gamma^5 q \ , \\ j_L^{a\,\mu} &= \bar{q}\gamma^\mu P_L t_L^a q \ , & j_R^{a\,\mu} &= \bar{q}\gamma^\mu P_R t_R^a q \ , \end{aligned} \tag{18}$$

where $t_L^a = t_R^a = \sigma^a/2$. It is straightforward to check whether these currents are anomalous with respect to the QCD $SU(3)_c$ gauge group. Doing so, one finds that $SU(2)_L$, $SU(2)_R$, and $U(1)_V$ are all anomaly free, while $U(1)_A$ is anomalous and therefore not a symmetry of the full quantum theory. Thus, we should aim to build a low-energy effective theory that is symmetric under $G_{flav} = SU(2)_L \times SU(2)_R \times U(1)_V$.

2.2 Symmetry Breaking and NGBs

Before attempting to construct such a theory, we should take note of a specific feature of confinement in this system (that has been verified in lattice studies): strong coupling in QCD generates a non-zero expectation value for the gauge invariant $\bar{q}q$ quark operator [7],

$$\langle \bar{q}_{R\,J} q_{L\,I} \rangle \ = \ \Lambda^3 \delta_{IJ} \ , \tag{19}$$

where $\Lambda \sim \Lambda_{QCD}$, and I and J run over u and d. This *quark condensate* expectation value *does not* respect the global symmetry group G_{flav}. In particular, applying a general G_{flav} transformation to the operator, the expectation value changes according to

$$\Lambda^3 \delta_{IJ} \ \to \ \Lambda^3 (LR^\dagger)_{IJ} \ . \tag{20}$$

Therefore the quark condensate spontaneously breaks G_{flav} to a smaller subgroup. It is not hard to see that this subgroup is $H_{flav} = SU(2)_V \times U(1)_V$, where $SU(2)_V$ is the subgroup of $SU(2)_L \times SU(2)_R$ transformations with $L = R$. The global G_{flav} symmetry is sometimes called a *chiral* symmetry, and its breaking is referred to as *chiral symmetry breaking*.

The spontaneous symmetry breaking pattern $G_{flav} \to H_{flav}$ has three broken generators, and there will be three corresponding massless Nambu-Goldstone bosons (NGBs). A generic expectation is that the other QCD excitations will pick up masses on the order of Λ_{QCD}, so let us begin by trying to write an EFT for the NGBs and worry about other possible states later. In this context, the unbroken $SU(2)_V$ symmetry is called *isospin*, while the unbroken $U(1)_V$ corresponds to baryon number (up to an overall normalization of the generators). Since chiral symmetry breaking plays an essential role in constructing this EFT, it is usually called *chiral perturbation theory*.

There is no unique way to build the EFT for the NGBs, but we should at least make sure that it respects the full underlying G_{flav} global symmetry, has three explicit degrees of freedom, and that the corresponding field excitations vanish in the vacuum configuration of the theory. A convenient way to accomplish these tasks is to use field variables that look like spacetime-dependent G_{flav}/H_{flav} transformations acting on the quark condensate vaccuum.[1]

[1]Recall that we did things like this when discussing NGBs in `notes-04` [10].

Comparing to Eq. (20), we note that $LR^\dagger$ is a 2×2 unitary matrix with unit determinant, and can be written as an exponential of Pauli matrices. This motivates building a theory in terms of the 2×2 matrix of fields [7, 11]

$$\Sigma(x) = \exp\left[2i\,\Pi^a(x)t^a/f\right] \ , \tag{21}$$

where $t^a = \sigma^a/2$, the $\Pi^a(x)$ are the dynamical NGB fields, and f is an as-yet unspecified parameter with dimension of mass. Also motivated by the picture of transformations acting on the vaccuum, we assume further that the field matrix transforms under G_{flav} by

$$\Sigma(x) \ \to \ L\,\Sigma(x)\,R^\dagger \ . \tag{22}$$

While the form of Eq. (21) might not be immediately obvious, it satisfies all the requirements for nice field variables (*i.e.* the right number of excitations (3), well-defined transformation properties, vanishing in the vaccuum) provided the vacuum of the EFT coincides with $\Sigma \to \mathbb{I}$.

The G_{flav} transformation rule of Eq. (22) defines how the Π^a fields transform under this group in a complicated way. To work this out, it is useful to write the transformation matrices in two equivalent ways:

$$\begin{aligned} L &= e^{ic_L^a t^a} & \qquad R &= e^{ic_R^a t^a} \\ &= e^{ic_A^a t^a} e^{ic_V^b t^b} & \qquad &= e^{-ic_A^a t^a} e^{ic_V^b t^b} \ . \end{aligned} \tag{23}$$

In the second form, we see that $SU(2)_V$ coincides with $c_A^a = 0$. Acting with an infinitesimal $SU(2)_V$ transformation on Σ ($L = R := V$) we find that

$$\Sigma(x) \ \to \ V\Sigma V^\dagger \ = \ \exp\left[2i\,V\Pi^a(x)t^aV^\dagger/f\right] \ , \tag{24}$$

which implies

$$\Pi^a \ \to \ \Pi'^a = \left(\delta^{ac} - f^{abc}c_V^b\right)\Pi^c + \mathcal{O}(c_V^2) \ . \tag{25}$$

Thus Π transforms linearly and in the adjoint representation of $SU(2)_V$. Transformations by the broken generators of G_{flav}/H_{flav} correspond to $c_V^a = 0$, and yield

$$\Pi^a \ \to \ \Pi'^a = \Pi^a + f c_A^a + \mathcal{O}(c_A^2) \ . \tag{26}$$

This is a non-linear transformation on Π^a, and it takes precisely the shift form we expect for a NFB field [10]. These nice transformation properties are the reason why the seemingly funny choice of field variables made in Eq. (21) is useful.

We turn next to building a Lagrangian in terms of the field variables Σ. We do not know how to do this exactly, but a necessary requirement is that it be symmetric under the full G_{flav} group, even part of the group is spontaneously broken. The simplest real and symmetric combination of Σ fields is not helpful:

$$\Sigma^\dagger\Sigma = \mathbb{I} = \Sigma\Sigma^\dagger \ . \tag{27}$$

To get something non-trivial, we need derivatives. The lowest-order terms are

$$\partial_\mu\Sigma^\dagger\partial^\mu\Sigma \to R(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma)R^\dagger \ , \qquad \partial_\mu\Sigma^\dagger\partial^\mu\Sigma \to L(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)L^\dagger \ , \tag{28}$$

implying that the trace $tr(\partial\Sigma^\dagger\cdot\partial\Sigma) = tr(\partial\Sigma\cdot\partial\Sigma^\dagger)$ is real and invariant under the symmetries. Thus, the leading term in the Lagrangian of the theory is

$$\begin{aligned} \mathscr{L}_{p^2} &= \frac{f^2}{4}\,tr(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma) \\ &= \frac{1}{2}(\partial\Pi^a)^2 - \frac{1}{3f^2}\left[(\Pi^a)^2(\partial\Pi^a)^2 + \ldots\right] + \ldots \end{aligned} \tag{29}$$

The first term is a canonical kinetic term for the Π fields while the second is a non-renormalizable interaction. Relative to the first term, the leading interaction is suppressed by a factor of p^2/f^2, and higher-order terms in the expansion come with even more powers of Π^a/f. Thus, this theory appears to be useful as an EFT for momenta that are small relative to f.

Terms of even higher order can be included as well, and they come with suppressions of at least p^4/f^4. For example [11],

$$\begin{aligned} \mathscr{L}_{p^4} = \;& L_1\left[tr(\partial\Sigma^\dagger\cdot\partial\Sigma)\right]^2 \\ &+ L_2\,tr(\partial_\mu\Sigma^\dagger\partial_\nu\Sigma)\,tr(\partial^\mu\Sigma^\dagger\partial^\nu\Sigma) \\ &+ L_3\,tr(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\,\partial_\nu\Sigma^\dagger\partial^\nu\Sigma)\ , \end{aligned} \tag{30}$$

where L_1, L_2, and L_3 are unknown dimensionless coupling constants.

2.3 Connecting NGBs with Reality

With a sensible EFT in hand, the next step is to connect the dynamical fields it contains to physical particles and fix the numerical value of f (and the other couplings). While there are no massless hadrons, there are three very light mesons, the π^0 and $\pi^\pm$. These are the particles we will connect with the Π^a fields in our EFT.

To make this connection more precise, let us extend the minimal two-flavour quark theory to include QED. This corresponds to gauging a $U(1)_{em}$ subgroup of G_{flav} corresponding to the combined generator

$$Q \;:=\; t_L^3 + t_R^3 + \frac{1}{6}\mathbb{I}\ . \tag{31}$$

Acting on q_L, we have $t_L^3 = \sigma^3/2$ and $t_R^3 = 0$ to give $Q = \sigma^3/2 + 1/6 = diag(2/3, -1/3)$, and similarly for q_R. Note as well that $U(1)_{em}$ is a subgroup of $H_{flav} = SU(2)_V \times U(1)_V$ that remains unbroken after confinement. Applying $U(1)_{em}$ transformations to the Π^a, as defined by the G_{flav} generator, we find that they can be arranged into combinations with well-defined electric charges $Q = 0,\ \pm 1$,

$$\pi^0 = \Pi^3, \qquad \pi^\pm = \frac{1}{\sqrt{2}}(\Pi^1 \mp i\Pi^2)\ . \tag{32}$$

We identify these states with the neutral and charged pions observed in nature.

To make further progress, we need to compare to data and fix the dimensionful parameter f in the theory. A nice way to do this is to match the Noether currents for G_{flav} in the quark theory with the corresponding currents in the EFT. Even though the choice of field variables is up to us, the currents correspond to physical charges and should be independent of the variables used to describe the system. We derived the currents in the quark theory in Eq. (18). In the EFT, applying Noether's theorem to the terms in Eq. (29) gives

$$\begin{aligned} j_V^\mu &= i\left(\pi^+\partial^\mu\pi^- - \pi^-\partial^\mu\pi^+\right) + \ldots \\ j_L^{a\,\mu} &= -i\frac{f^2}{2}tr(\Sigma^\dagger t^a\partial^\mu\Sigma) = f\,tr(t^a\partial^\mu\Pi) + \mathcal{O}(\Pi^2)\ , \\ j_R^{a\,\mu} &= -i\frac{f^2}{2}tr(\Sigma\, t^a\partial^\mu\Sigma^\dagger) = -f\,tr(t^a\partial^\mu\,\Pi) + \mathcal{O}(\Pi^2)\ . \end{aligned} \tag{33}$$

Consider now the decay of a negatively charged pion to a muon and an antimuon neutrino. This proceeds through a W^-, and its amplitude is proportional to the matrix element [7]

$$\langle\mu^-\bar{\nu}_\mu\,|H_{int}|\,\pi^-(p)\rangle\ , \tag{34}$$

with the interaction operator given by

$$H_{int} = \frac{4G_F}{\sqrt{2}}(\bar{u}\gamma^\mu P_L d)\,(\bar{\mu}\gamma_\mu P_L\nu_\mu)\ . \tag{35}$$

Contracting fields with external states, the matrix element factorizes into a simple leptonic piece, and a complicated hadronic piece given by

$$\langle 0\,|\bar{u}\gamma^\mu P_L d\,|\pi^-(p)\rangle := -i\frac{1}{\sqrt{2}}\,f_\pi p^\mu\ , \tag{36}$$

where the right-hand side is fixed by Lorentz invariance. Now, we can write this quark operator in terms of a current

$$\bar{u}\gamma^\mu P_L d = \left(j_L^{1\,\mu} + i\,j_L^{2\,\mu}\right) = \frac{1}{\sqrt{2}}\,f\,\partial^\mu\pi^- + \mathcal{O}(\pi^2)\ . \tag{37}$$

Plunking this into the pion matrix element, we see that to leading order

$$f = f_\pi\ \simeq\ 92\ \text{MeV}\ , \tag{38}$$

where the latter numerical value is extracted from the measured the rate of pion decays.[2] Measurements of pion scattering can be used in a similar way to fix the values of L_1, L_2, and L_3.

2.4 Explicit Symmetry Breaking and Masses

In the theory so far, the Π^a fields that we identify with physical pions are massless. This is in obvious disagreement with the observed pion masses of [3]

$$m_{\pi^0} = 139.57\ \text{MeV}\ , \qquad m_{\pi^\pm} = 134.98\ \text{MeV} \tag{39}$$

[2]Note that other conventions exist for f_π that differ by factors of $\sqrt{2}$.

The reason for this disagreement is that some important physics has been left out of the effective theory. Specifically, part of the global G_{flav} symmetry is broken explicitly by the non-zero u and d quark masses (as well as gauging $U(1)_{em}$). Fortunately, these masses are very small compared to the other dimensionful parameter in the EFT, $f = f_\pi$, and they can be treated systematically as small perturbations.

To add the quark masses, it is convenient to write them in the matrix form

$$-\mathscr{L} \supset \bar{q}_R M q_L + \bar{q}_L M^\dagger q_R \; . \tag{40}$$

This term breaks $SU(2)_L \times SU(2)_R$ down to $SU(2)_V$ for $m_u = m_d$, and to nothing for $m_u \neq m_d$. To keep track of the breaking, it is helpful to notice that the full G_{flav} symmetry would be preserved is the fixed mass matrix also transformed according to

$$M \rightarrow RML^\dagger \; . \tag{41}$$

Even though M does not actually transform in this way, the EFT should exhibit the full symmetry if we pretend it does. The leading correction to the EFT that can be written with this in mind is [7]

$$-\mathscr{L} \supset \frac{1}{2}\tilde{\Lambda}^3\, tr(M\Sigma) + h.c. \tag{42}$$

where we expect $\tilde{\Lambda} \sim \Lambda$. Expanding this out, one obtains a pion mass term of

$$m_\pi^2 f_\pi^2 = \tilde{\Lambda}^3(m_u + m_d) \; . \tag{43}$$

As expected, the pion masses go to zero as the underlying quark masses vanish since the approximate G_{flav} symmetry becomes exact.

There is an additional mass splitting between the pions due to QED effects [7]. This arises because QED gauges only a subgroup of the whole G_{flav} global symmetry, and therefore represents an additional source of explicit breaking of the flavour group. Relative to the dominant QCD dynamics, QED effects are much weaker and we can treat them as small perturbations on the picture we have derived. Indeed, the correction to the squared masses due to electromagnetism goes like $\alpha_{em} f_\pi^2$ and is subleading compared to the effects of the explicit masses. This is the main source of the small mass splitting between the neutral and charged pions. In addition, the flavour group G_{flav} is anomalous with respect to QED, and this leads to a coupling between the π^0 and two photons that provides the dominant decay channel of the neutral pion with a branching fraction of nearly 99%.

2.5 Adding the Strange Quark

Up to now we have neglected the strange quark, but it turns out to be a pretty good approximation to include it in the NGB description as well and treat its mass as another small perturbation. The theory now has an approximate $SU(3)_L \times SU(3)_R \times U(1)_V$ global symmetry that is spontaneously broken by the QCD vacuum down to $SU(3)_V \times U(1)_V$. To derive a low-energy EFT, we repeat the steps above but now with a 3×3 field of the form

$$\Sigma = \exp\left[2i\Pi^a(x)t_3^a/f\right] \ , \tag{44}$$

where the eight matrices t_3^a generate the fundamental of $SU(3)$. This expansion yield an octet of eight (pseudo-) NGBs that can be identified with the pions and kaons. More precisely, the components of the expanded Σ field correspond to [7]

$$\Pi^a t^a = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc} \pi^0/\sqrt{2}+\eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2}+\eta/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta/\sqrt{6} \end{array}\right) \ . \tag{45}$$

As before, we can derive the approximate masses of these states by adding the 3×3 mass matrix $M = diag(m_u, m_d, m_s)$ to the theory as a small perturbation. These masses agree pretty well with the observed values. A very different set of technology is needed to describe mesons involving c and b quarks. The top quark, being very heavy, decays too quickly to form meson bound states.

2.6 Baryons

This framework can also be used to describe baryons, such as the proton and neutron [7]. Their defining feature in this context is that they carry a net charge under the $U(1)_V$ subgroup of G_{flav} (and H_{flav}).[3] These states do not correspond to NGBs, and have to added to the theory by hand. Still, we can constrain the EFT for them by demanding that the corresponding Lagrangian terms respect the G_{flav} symmetry.

Treating baryons in this EFT framework might seem a little puzzling because their masses are at least as large as $m_p \simeq 938$ MeV, and much greater than f_π. According to our previous EFT discussion, it seems like we should integrate them out. The reason for keeping them in the theory is that the lightest among them, the proton and neutron (to a good approximation), are stable. This implies that we can prepare experiments in which protons and neutrons scatter with each other with low initial and final momenta relative to their mass, and thus our EFT can be applied within its range of validity to describe such processes. Such scattering can also create heavier baryons (which are relatively close in mass to the nucleons), so we should include these states in our theory as well. In contrast, our low-energy QCD EFT would not be useful for calculating the annihilation of a proton with an anti-proton, since now the particles in the final state have typical momenta on the order of $p \simeq m_p \sim$ GeV.

2.7 Beyond NGBs

In addition to the NBG-like mesons, QCD confinement also produces many other meson states that do not correspond to NGBs. These appear as resonances in $e^+e^- \to$ hadrons,

[3] For this reason, $U(1)_V$ corresponds to *baryon number*, up to a normalization factor.

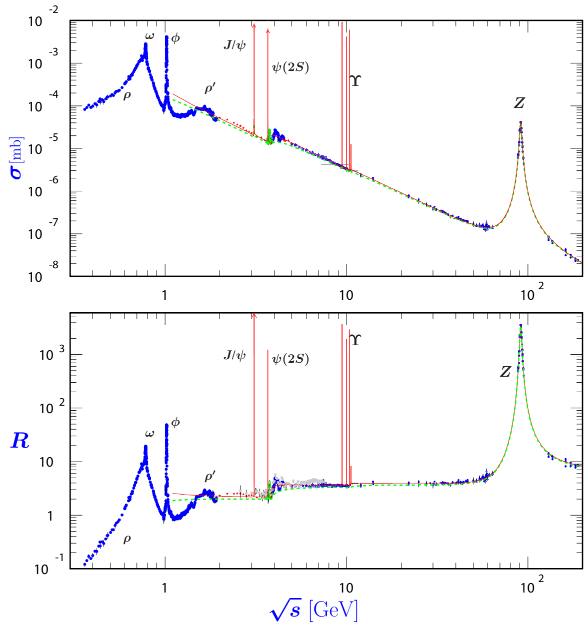

Figure 1: Total cross section and ratio $R(s)$ for inclusive hadronic production in e^+e^- collisions as functions of the CM energy $\sqrt{s}$, from Ref. [3].

as illustrated in Fig. 1, taken from Ref. [3]. In the upper panel of the figure we show the hadronic production cross section as a function of $\sqrt{s} = \sqrt{(p_1 + p_2)^2}$, while in the lower panel we have the ratio $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. The region that is the most challenging to describe theoretically is $\sqrt{s} \in [500\text{ MeV}, 2\text{ GeV}]$. Here, neither chiral perturbation theory nor perturbative QCD (in terms of quarks and gluons) works very well. We also see a broad resonance in this region in Fig. 1 corresponding to the ρ meson ($m_\rho \simeq 770$ MeV); the large breadth implies that it couples strongly to other modes and therefore plays a very important role in this energy range.

The appearance of new non-NGB meson states coincides with the breakdown of chiral perturbation theory since they were not included in the EFT. Recall that in expanding the leading EFT operator of Eq. (29), we found an infinite tower of operators with two derivatives, multiple Π^a fields, and suppression by factors of $f \simeq 93$ MeV. Naïvely, this would seem to imply that the EFT is only valid up to $p \ll f$. However, it turns out that chiral perturbation theory is reliable up to momenta considerably higher than this, on the order of $p \sim 4\pi f \sim m_\rho$. One reason to expect that this might work is that the scale f does not correspond to the appearance of any new non-NGB physics.

To estimate the full range of chiral perturbation theory, let us start with only the leading operator of Eq. (29) and study the sizes of new operators are generated by loops. The leading interaction from Eq. (29) has the form $(\pi\partial\pi)^2/f^2$ and corresponds to a 4-point momentum-space vertex of the form

$$V_2 \sim \frac{p^2}{f^2} . \tag{46}$$

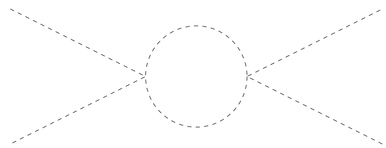

Figure 2: Divergent loop diagram with two V_2 vertices that generates a V_4 vertex.

Loops involving this (and other vertices from Eq. (29)) will renormalize the operator of Eq. (29), but they will not change its form due to the underlying symmetries of the theory. However, loops involving multiple V_2 vertices will generate new operators with more powers of momenta. For example, combining a pair of V_2 vertices in a loop as shown in Fig. 2 produces a log-divergent Π-field 4-point function of the form

$$\Delta\Gamma^{(\pi^4)} \sim \frac{1}{(4\pi)^2}\frac{p^4}{f^4}\ln\left(\frac{\Lambda^2}{p^2}\right) , \tag{47}$$

where we have cut off the internal loop momentum at $q_E^2 = \Lambda^2$ and the $1/(4\pi)^2$ factor is the standard dimensionless contribution from each loop. The form of the correction corresponds to an operator with four derivatives, which by symmetry must take the form of those listed in Eq. (30) with dimensionless coefficient $L_i \sim 1/(4\pi)^2$. The induced 4-point vertex from these new operators therefore takes the form

$$V_4 \sim \frac{p^2}{f^2}\left(\frac{p}{4\pi f}\right)^2 . \tag{48}$$

More generally, symmetric operators with $2n$ derivatives are generated at $(n-1)$-loop order, and these give contribution to the 4-point vertex of [7, 8]

$$V_{2n} \sim \frac{p^2}{f^2}\left(\frac{p}{4\pi f}\right)^{2n-2} . \tag{49}$$

A similar counting applies to other types of vertices.

This analysis shows that the initial non-renormalizable operator of Eq. (29) generates an infinite number of new operators, as expected, but with a well-defined parametric form. For our EFT to useful, the effects of operators of increasingly higher dimension, with more derivatives ($2n$) in this case, must fall off with increasing n. Based on the scaling of Eq. (49), we see that the necessary condition is

$$p \ll 4\pi f . \tag{50}$$

The range of validity can therefore be parametrically larger than f. Indeed, it is standard practice to write the cutoff of the NGB EFT as $\Lambda_\chi \sim 4\pi f$.

Comparing to experimental data, this counting of loops and $(4\pi)^2$ factors actually seems to work. For example, the measured values of the dimensionless coefficients L_i defined in Eq. (30) are on the order of $L_i \sim (\text{few}) \times 10^{-3} \sim 1/(4\pi)^2$ [11]. The breakdown scale of the

NGB EFT near Λ_χ also coincides with the appearance of new physics in the form of the ρ meson, with $m_\rho \simeq 770$ MeV. This is consistent with the estimate based on Eq. (49) and only slighly below the maximal value.

To put this another way, the appearance of increasingly higher-dimensional operators coincides with them being generated by integrating out heavy modes such as the ρ meson. Related to this, it is standard practice to define

$$g_\rho = m_\rho/f \ . \tag{51}$$

The dimensionless ratio $g_\rho > 1$ is interpreted as the effective coupling of the ρ (and other non-NGB states) to the NGBs, and its size reflects that the system is strongly coupled. To see why, note that the minimal interaction between the vector ρ and the pions should have the schematic form

$$-\mathscr{L} \ \supset \ g_\rho \left(\pi\partial_\mu\pi\right)\rho^\mu - \frac{1}{2}m_\rho^2\rho_\mu\rho^\mu \ . \tag{52}$$

Integrating out the massive ρ at tree level generates effective pion interactions of the form

$$-\mathscr{L} \ \supset \frac{g_\rho^2}{m_\rho^2}\left[(\pi\partial\pi)^2 + \frac{1}{m_\rho^2}(\pi\partial^2\pi)^2 + \ldots\right] \ , \tag{53}$$

where the first term is the leading piece and the addtional terms are higher orders in the expansion of the ρ propagator in powers of p^2/m_ρ^2. Comparing to our previous results, the first term matches the form of the interaction in Eq. (29) using $1/f^2 = g_\rho^2/m_\rho^2$ while the second agrees with the leading interaction in Eq. (30) after identifying $L_i \sim (f/m_\rho)^2 \sim 1/(4\pi)^2$. Going to higher orders in derivatives, the operator scaling factors match the power counting of loops discussed above.

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