P528 Notes #8: How To Think About QFTs

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Chances are you are fairly new to quantum field theory (QFT) [1, 2, 3]. It is a rich and complicated topic that can take many years to get an intuitive feel for. In these notes we will try to accelarate the process by describing the "modern" interpretation of QFT that is used widely in studies of the SM and beyond. The main aspects of this are dimensional analysis, symmetries, renormalization, and effective field theories.

1 Dimensional Analysis

Dimensional analysis (DA) is an incredibly powerful tool for estimating the sizes of things without doing any hard calculations. The idea is to keep track of the mass dimensions of everything in the problem.[1] For DA to work optimally, it is also essential to keep track of the exact and approximate symmetries of the theory. We will illustrate this below with example.

Recall that we usually define a QFT (such as the SM) with an action in d spacetime dimensions. Consider a theory with a scalar ϕ and a fermion ψ with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \overline{\psi}i\gamma^\mu\partial_\mu\psi - M\overline{\psi}\psi - \frac{A}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 - y\,\phi\,\overline{\psi}\psi \tag{1}$$

To figure out the mass dimensions, we use the fact that the action $S = \int d^dx\ \mathscr{L}$ is always dimensionless. Using a square bracket to denote the mass dimension, we also have

$$[S] = 0, \quad [x] = -1, \quad [\partial] = +1, \quad [d^dx] = -d\ . \tag{2}$$

Each term making up the action must be separately dimensionless. Applying this to the scalar and fermion kinetic terms, we find

$$0 \ = \ -d + 2[\phi] + 2 \ = \ -d + 2[\psi] + 1 \tag{3}$$

so that we have

$$[\phi] = 1 + \frac{(d-4)}{2}, \quad [\psi] = 3/2 + \frac{(d-4)}{2}\ . \tag{4}$$

We will mostly work in $d = 4$, giving $[\phi] = 1$ and $[\psi] = 3/2$. Moving on to the other terms in the Lagrangian, we find

$$[m] = [M] = 1, \quad [A] = 1 + \frac{(4-d)}{2}, \quad [\lambda] = 0 + (4-d), \quad [y] = 0 + \frac{(4-d)}{2}\ . \tag{5}$$

Note that the "mass terms" always have dimensions of mass.

[1] Since we are using natural units with $\hbar = c = 1$, everything can be expressed with dimensions of mass.

As a first application of dimensional analysis, suppose we have $m \gg M$, so that the decay $\phi \to \psi\bar{\psi}$ is possible, and let us estimate the corresponding decay rate. In the limit $y \to 0$ the scalar and the fermions would not interact at all in this theory, and thus least one power of y is needed in the decay amplitude. With $m \gg M$, the outgoing ψ particles will be highly relativistic with energies on the order of m. This implies that m is the only relevant dimensionful quantity for the decay in this limit. Since the decay rate has a mass dimension of one (equivalent to 1/time), the estimate from DA is

$$\Gamma \sim y^2 m \ . \tag{6}$$

Up to factors of two and 4π and kinematic corrections that depend on $M/m \ll 1$, this is a good estimate of the full tree-level calculation.

Consider next $\phi\phi \to \phi\phi$ scattering in the limit $y \to 0,\ A \to 0,\ g \to 0$. This process can now occur only through the $\lambda\phi^4$ operator, so the scattering amplitude requires at least one power of λ. At very high energies, the only relevant dimensionful quantity is the centre-of-mass energy $s = (p_1 + p_2)^2 \gg m^2 = 4E_{cm}^2$. Since the total scattering cross section has a mass dimension of minus two (equivalent to area), the DA estimate is

$$\sigma \sim \frac{\lambda^2}{s} \ . \tag{7}$$

This is a reasonable approximation to the full high-energy result.

If the cubic coupling A is non-zero, there is another contribution to the $\phi\phi \to \phi\phi$ scattering cross section. Two powers of the coupling are needed to make an amplitude with an even number of external states. Taking into account that A has a mass dimension of one, its contribution to the cross section (neglecting interference with the λ piece) is

$$\sigma \sim \frac{A^4}{s^3} \ . \tag{8}$$

This falls off more quickly with energy than the contribution from λ.

2 Renormalization

Calculations in QFT involving Feynman diagrams with loops often lead to apparent divergences. This can be unsettling at first, but it turns out that such divergences have interesting physical implications. Before getting into a few explicit examples, let us mention that divergences can arise both at very low energies (equivalent to large distances – called the IR limit) and at very high energies (equivalent to short distances – called the UV limit). We will concentrate on UV divergences here, since IR divergences often cancel out after taking into account the finite resolution of experiments and including in the calculation diagrams with additional external legs [1, 2, 3].

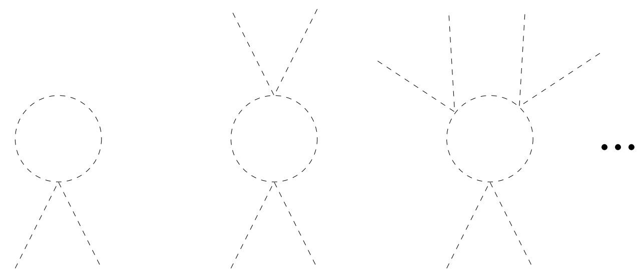

Figure 1: Corrections to 1PI n-point functions of the $\lambda\phi^4$ theory at one-loop order.

2.1 Basic Renormalization

To discuss renormalization, consider a real scalar theory with the Lagrangian [1]

$$\mathscr{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4\ . \tag{9}$$

In $d = 4$, λ_0 is dimensionless and m_0 has dimensions of mass. These terms give a propagator of $i/(p^2 - m_0^2 + i\epsilon)$ and a 4-point vertex $-i\lambda_0$. The basic objects to be computed in this theory are $i\Gamma^{(n)}$, the 1PI connected n-point functions in momentum space with all external propagators removed. At tree-level, $i\Gamma^{(2)}$ is the usual propogator, $i\Gamma^{(4)}$ is the 4-point vertex, and all others vanish. At one-loop order there are loop corrections to the $(2n)$-point functions, shown in Fig. 1. Ignoring external momenta, these corrections go like $(n \geq 1)$

$$\begin{aligned}\Delta\Gamma^{(2n)} &\sim \lambda_0^n \int \frac{d^4q}{(2\pi)^4}\left(\frac{1}{q^2 - m_0^2 + i\epsilon}\right)^n \\ &\sim \lambda_0^n \lim_{\Lambda\to\infty}\int d\Omega_{\hat{q}_E}\int_0^\Lambda dq_E\, q_E^3\left(\frac{1}{q_E^2 + m_0^2}\right)^n \\ &\sim (\text{finite}) + \lim_{\Lambda\to\infty}\Lambda^{(4-2n)}\ ,\end{aligned} \tag{10}$$

where we have transformed to Euclidean space with $q^0 = iq_E^0$ and $q_E^2 = (q_E^0)^2 + \vec{q}^{\,2}$ and $\Omega_{\hat{q}_E}$ is the solid angle in this 4-dimensional space. Note as well that $\Gamma^{(n)} = (\text{tree-level}) + \Delta\Gamma^{(n)}$. These integrals diverge in the UV ($q_E \to \infty$) for $n = 0$, 1, 2, but are convergent for all higher n (and "Λ^0" diverges as $\ln\Lambda$).

What are we to do with these seemingly infinite quantum corrections to the theory? The divergences come from loops in which we sum over all possible momenta. In other words, we are assuming implicitly that our theory is valid up to arbitrarily high momenta, well above what can be accessed experimentally. This suggests that we are asking more of the theory than what it can provide.[2] Instead, let us treat the theory as an *effective theory* valid up to

[2] Note that something similar happens in classical electrodynamics (or Newtonian gravity) when one tries to construct a point charge. The energy required to build a uniform charge distribution of radius R and total charge Q is proportional to Q^2/R, which obviously diverges as $R \to 0$.

some very high energy scale where we assume new physics comes in and makes the integrals finite. At first glance, this does not look much better; we have traded formal infinities for unknown finite quantitites and it is not immediately obvious how this will help us make testable predictions. However, it turns out that in a *renormalizable* theory the effects of the unknown new physics can be parametrized by a finite set of unknown coefficients in the low-energy theory. Once these unknown coefficients are fixed using experimental data, everything else computed in the theory is a genuine prediction.

Let us illustrate this in the scalar theory. We will use a method called *renormalized perturbation theory* following the treatment of Ref. [1]. The first step is to rewrite the Lagrangian in terms of the rescaled field $\phi_0 = Z^{1/2}\phi$. Plugging into the original Lagrangian and rearranging, we get

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi^4 & (11) \\ &= \frac{1}{2}Z(\partial\phi)^2 - \frac{1}{2}Zm_0^2\phi^2 - \frac{\lambda_0}{4!}Z^2\phi^4 & (12) \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 & (13) \\ &\qquad + \frac{1}{2}\delta Z(\partial\phi)^2 - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \ , \end{aligned}$$

where

$$\delta Z = (Z-1)\ , \quad \delta m^2 = Zm_0^2 - m^2\ , \quad \delta\lambda = Z^2\lambda_0 - \lambda\ . \tag{14}$$

The coefficients in the last line of Eq. (12) are called *counterterms*.

Using this rearranged Lagrangian, let us go back and calculate various n-point functions (for ϕ). The strategy will be to compute with the terms in the first line of Eq. (12), and then add the counterterms as small corrections that begin at one-loop order (so that at tree level $Z = 1$, $\delta m^2 = 0 = \delta\lambda$). The propagator of the theory is now $i/(p^2 - m^2 + i\epsilon)$ and the vertex is $-i\lambda$. We also have counterterm corrections in the form of a 2-point interaction $i(\delta Z\, p^2 - \delta m^2)$ and a 4-point interaction $-i\,\delta\lambda$.

With these interactions and our new interpretation of the theory, the renormalization process involves two steps. The first is to *regulate* the would-be divergent integrals to make them finite. There are many ways to do this, but for now we will simply transform to Euclidean space $q^0 = iq_E^0$ and impose an upper cutoff on the Euclidean magnitude of

$$q_E^2 = (q_E^0)^2 + \vec{q}^{\,2} \leq \Lambda^2\ . \tag{15}$$

The one-loop correction to the 2-point function becomes (schematically)

$$\begin{aligned} i\Delta\Gamma^{(2)}(p^2) &= -i\,\frac{\lambda}{2}\int^{\Lambda}\frac{d^4q_E}{(2\pi)^4}\left(\frac{1}{q_E^2+m^2}\right) \;+\; i(p^2\delta Z - \delta m^2) & (16) \\ &= ip^2\left[\lambda A_1\ln\left(\frac{\Lambda^2}{a_pp^2+a_mm^2}\right) + \lambda A_2 + \delta Z\right] \\ &\qquad - i\left[\lambda B_0\Lambda^2 + \lambda B_1 m^2\ln\left(\frac{\Lambda^2}{b_pp^2+b_mm^2}\right) + \lambda B_2m^2 + \delta m^2\right]\ , \end{aligned}$$

where the coefficients A_i, B_i, and a_i are finite and dimensionless. The form of this result is dictated by dimensional analysis together with counting the degrees of divergence and the number of coupling factors. Similarly, the one-loop 4-point function takes the schematic form

$$\begin{aligned} i\Gamma^{(4)}(p^2) &= -i\lambda \;+\; i\lambda^2 \int^{\Lambda} \frac{d^4 q_E}{(2\pi)^4} \left(\frac{1}{q_E^2 + \Delta}\right)^2 - i\delta\lambda \\ &= -i\lambda + i\lambda^2 \left[C_1 \ln\left(\frac{\Lambda^2}{c_p p^2 + c_m m^2}\right) + C_2\right] - i\delta\lambda \;, \end{aligned} \tag{17}$$

where p^2 is the momentum scale that characterizes the process of interest, Δ is a function of p^2 and m^2 of mass dimension two, and C_1, C_2, and c_i are finite and dimensionless. At one-loop, all the other 1PI connected n-point functions are finite.

The second step is the renormalization part itself. This amounts to fixing the counterterms by imposing *renormalization conditions* that relate the parameters m^2 and λ to experimental observables [1, 4]. Here, we need three renormalization conditions to fix the three counterterms, and these will require two experimental inputs and one convention choice for normalizing the propagator.

Starting with the 2-point function, recall that the pole of the resummed propagator is the physical mass of the particle corresponding to the field ϕ:

$$(\text{Prop}) \;=\; \frac{iR}{p^2 - m^2_{phys}} + (\text{non-singular}) \;, \tag{18}$$

where R is the residue of the pole. Given this fact, a popular choice of renormalization conditions to fix is to identify m^2 with the measured particle mass, and to demand that the residue of the pole of the propagator is unity. Using our 1-loop result, the resummed propagator is

$$\begin{aligned} (\text{Prop}) &= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(i\Delta\Gamma^{(2)})\frac{i}{p^2 - m^2} + \ldots \\ &= \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left(\frac{-\Delta\Gamma^{(2)}}{p^2 - m^2}\right)^n \\ &= \frac{i}{p^2 - m^2 + \Delta\Gamma^{(2)}(p^2)} \;. \end{aligned} \tag{19}$$

Applying the two renormalization conditions by expanding around $p^2 = m^2 = m^2_{phys}$,

$$0 \;=\; \Delta\Gamma^{(2)}\Big|_{p^2=m^2} \;, \tag{20}$$

$$0 \;=\; \frac{d\Delta\Gamma^{(2)}}{dp^2}\bigg|_{p^2=m^2} \;. \tag{21}$$

This gives two equations in two unknowns that allow us to solve for δZ and δm^2. Note that the second condition above is chosent to enforce $R = 1$ at the $p^2 = m^2$ pole.

We also need to deal with $\delta\lambda$. A reasonable (but not unique) choice is to set the higher-order corrections to the two-to-two scattering amplitude to zero at the fixed momentum point $p^2 = m^2$. This corresponds to

$$0 = \Delta\Gamma^{(4)}(m^2) \ := \ \lambda + \Gamma^{(4)}(m^2) \ , \tag{22}$$

which fixes $\delta\lambda$ and relates λ directly to an observable cross-section.

Together, we have used the three renormalization conditions of Eqs. (20,21,22) to fix the parameters in the Lagrangian and determine the counterterms to one-loop accuracy. Including these counterterms, any other process computed to one-loop will be finite and provide an unambiguous prediction of the theory. For example, we can now predict two-to-two scattering cross sections for any other momentum values. The corresponding one-loop amplitude is

$$-i\mathcal{M} = -i\lambda + i\lambda^2 C_1 \ln\left(\frac{c_p p^2 + c_m m^2}{c_p m^2 + c_m m^2}\right) \ . \tag{23}$$

All the dependence of the theory on unknown UV physics has been absorbed into the finite parameters λ and m^2, which we have fixed in terms of observables. This is the cost of renormalization: we cannot predict observables starting only from the original *bare* parameters m_0^2 and λ_0^2 in Eq. (9). Instead, we are only able to predict observables in terms of a finite set of basis observables. Note as well that the renormalization conditions we chose are not unique – other choices are possible, and they would lead to a different relationship between the renormalized parameters m^2 and λ and observables.

2.2 Renormalizabilitly

The $\lambda\phi^4$ theory considered here is said to be *renormalizable*. This means that the renormalization picture can be extended to higher loop orders using the same finite set of counterterm interactions and renormalization conditions. That one can do this is related to the fact that only a finite number of n-point functions are UV-divergent (when organized in a sensible way) [3, 4].

A rough argument for renormalizability can be made based on dimensional analysis. In our $\lambda\phi^4$ theory consider what would be needed to obtain a UV divergence in the 6-point function $\Delta\Gamma^{(6)}$. By dimensional analysis this 6-point function would have to have a mass dimension of minus two. But to make this up, the only dimensionful quantities we have to work with are m^2, p^2, and Λ. However, we expect to have reasonably smooth limits as $m^2 \to 0$ or $p^2 \to 0$, so they should only appear as positive powers. This leaves Λ as the only quantity that can appear in the denominator, but negative powers of Λ do not correspond to UV divergences (as $\Lambda \to \infty$). Thus, the only divergences for which we need counterterms for appear in the 2- and 4-point (and 0-point) functions, and the counterterms we already have are enough for this.

More generally, *theories with couplings that have exclusively non-negative mass dimensions are typically renormalizable.* In a renormalizable theory, a finite number of experimental inputs are needed to fix a finite number of parameters of the theory (and define the

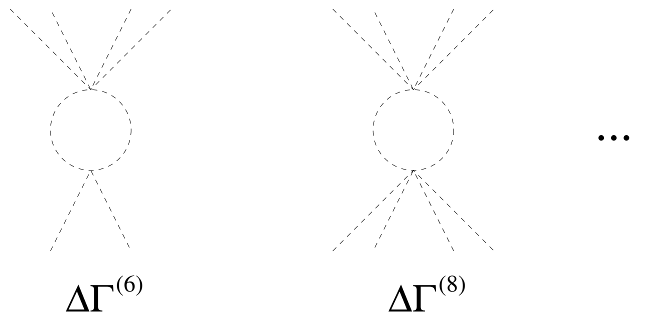

Figure 2: Corrections to ($n \geq 6$)-point functions when the $\zeta\phi^6/M^2$ operator is added.

counterterms along the way). The output of such a theory are predictions of many output observables in terms of a finite number of input observables. An important example of a renormalizable theory is the Standard Model (SM)!

But what about non-renormalizable theories? In such theories, an infinite number of experimental inputs are needed to fix an infinite number of independent divergences (as we will illustrate). While this sounds bad, non-renormalizable theories can still be useful and predictive provided we only use them at sufficiently low energies and compute to a finite accuracy [5, 6].

To illustrate this, let us extend the the scalar theory of Eq. (9) with a *higher-dimensional interaction*:

$$\mathscr{L} \;\; \to \;\; \mathscr{L} - \frac{\zeta}{M^2}\,\phi^6 \;, \tag{24}$$

where ζ is dimensionless and M has dimensions of mass. With this term, we now have a dimensionful quantity that can appear in denominators, and this allows the possibility of Λ appearing in numerators (or logarithms) of more n-point functions. For example, the first diagram in Fig. 2 goes like

$$\Delta\Gamma^{(6)} \;\sim\; \frac{\lambda\zeta}{M^2}\int\frac{d^4q}{(2\pi)^4}\left(\frac{1}{q^2-m^2+i\epsilon}\right)^2 \;\sim\; \frac{\lambda\zeta}{M^2}\,\ln\Lambda \tag{25}$$

This is a divergent correction to the 6-point function that needs a new counterterm of the form $\delta\zeta\,\phi^6/M^2$ to cancel off. The ϕ^6 interaction also produces new divergences in the 8-point function:

$$\Delta\Gamma^{(8)} \;\sim\; \frac{\zeta^2}{M^4}\,\ln\Lambda\;. \tag{26}$$

Going beyond one-loop order, the single ϕ^6 interaction of Eq. (24) produces an infinite number of new divergences. Each of them requires a counterterm and an experimental observable to fix its value. Since the number of input observables required is infinite, the theory is no longer renormalizable.

Things look bad at this point, but once again there is a way out of the mess provided we stick to energies much lower than the new dimensionful scale M. Consider the contribution of the ϕ^6 operator of Eq. (24) to a $2 \to 4$ scattering process with a characteristic momentum scale p^2. By dimensional analysis, the contribution to the cross section must scale like

$$\Delta\sigma \;\sim\; \zeta^2 \left(\frac{p}{M}\right)^4 \frac{1}{p^2} \;. \tag{27}$$

Contributions from operators of higher dimension contain even more powers of (p^2/M^2). In comparison, the contribution from the renormalizable $\lambda\phi^4$ operator goes like

$$\Delta\sigma \;\sim\; \lambda^2 \frac{1}{p^2} \;. \tag{28}$$

The key feature is that if we focus exclusively on processes at low energies $p^2 \ll M^2$ and demand predictions of finite accuracy, only a finite number of the higher-dimensional operators need to be considered. This means that we only need a finite number of experimental inputs, and we can make testable predictions for other observables with a well-defined theoretical uncertainty. On the other hand, the non-renormalizable theory becomes less and less useful as the characteristic energy scale p^2 approaches M^2 since more and more operators need to be included. When we discuss effective field theories below, we will argue that the dimensionful scale M can be identified with a mass scale of new physics.

2.3 Running Parameters and RG

Let us now go back to the basic renormalizable $\lambda\phi^4$ theory and examine renormalization in a slightly different way. The renormalization conditions of Eqs. (20,21,22) connect the renormalized parameters in this scheme in a direct way to physical observables. However, these renormalized parameters can be cumbersome to work with in practice. Instead, suppose we define renormalized parameters in a different way by choosing counterterms that remove the dependence on Λ in a minimal way. Specifically, let us choose

$$\delta Z \;=\; \lambda\, A_1 \ln\left(\frac{\mu^2}{\Lambda^2}\right) \tag{29}$$

$$\delta m^2 \;=\; -\lambda\, B_0 \Lambda^2 + \lambda\, B_1 m^2 \ln\left(\frac{\mu^2}{\Lambda^2}\right) \tag{30}$$

$$\delta\lambda \;=\; -\lambda^2\, C_1 \ln\left(\frac{\mu^2}{\Lambda^2}\right) \;, \tag{31}$$

where μ is an unspecified *renormalization mass scale* with no particular physical significance. Plugging these counterterms into Eqs. (16,17), it is not hard to see that they remove the cutoff dependence from $\Delta\Gamma^{(2)}$ and $\Delta\Gamma^{(4)}$ and make them “finite” as $\Lambda \to \infty$. For obvious reasons, this choice of counterterms is a form of what is called *minimal subtraction* (MS) [3, 4].

By choosing our counterterms in this minimal form, we have defined a **different** set of renormalized parameters $m^2(\mu)$ and $\lambda(\mu)$. They depend on the mass scale μ, and their connection to physical observables is more complicated than before.[3] Even so, we can still

[3]For example, $m^2(\mu)$ is no longer the physical particle mass.

fix their values at a given reference value $\mu = \mu_0$ by computing with them and matching to physical observables. The important advantage of this scheme is that we can choose μ to be whatever we like.

To see why this is useful, consider the loop-corrected amplitude for $\phi\phi \to \phi\phi$ scattering given in Eq. (23) derived for our previous *physical* choice of counterterms. As the scattering energy becomes large, with $p^2 \gg m^2$, the logarithm in the loop correction can embiggen. If such logarithms get big enough, they can overwhelm the suppression of the additional powers of couplings in the loop corrections, and perturbation theory breaks down even though the underlying renormalized coupling is small. In contrast, in our minimal renormalization scheme the one-loop scattering amplitude for characteristic momentum p^2 is related to the renormalized parameter $\lambda(\mu)$ by

$$i\Gamma^{(4)}(p^2) = -i\lambda(\mu) + i\lambda^2(\mu)\left[C_1 \ln\left(\frac{\mu^2}{c_p p^2 + c_m m^2}\right) + C_2\right] . \tag{32}$$

By choosing $\mu \sim \sqrt{|p^2|}$, the logarithm will be of order unity, and perturbation theory will be optimized. This relation also shows that $\lambda(\mu)$ is perturbatively close to the renormalized coupling we would derive in the physical scheme by choosing our renormalization condition for $\delta\lambda$ at the momentum scale p^2 of interest.

It is interesting to study how the renormalized parameters change as μ is varied. Going back to our original definitions of these couplings in terms of the bare couplings, Eq. (14), we can derive differential equations for this evolution [2, 7]. Starting with $1 = (Z - \delta Z)$, we find

$$\begin{aligned} \frac{1}{Z}\frac{dZ}{dt} &= \frac{1}{Z}\frac{d}{dt}(\delta Z) \\ &= 2\lambda\, A_1 + \mathcal{O}(\lambda^2) , \end{aligned} \tag{33}$$

where $t = \ln(\mu/\mu_0)$ for some reference scale μ_0. Note that since we only computed to one-loop order, we can only trust this result to leading non-trivial order in the coupling λ and we should discard the $\mathcal{O}(\lambda^2)$ piece. Next, $\lambda_0 = (\lambda + \delta\lambda)/Z^2$ is independent of μ, which implies

$$\begin{aligned} \frac{d\lambda}{dt} &= -\frac{d}{dt}(\delta\lambda) + \frac{2}{Z}\frac{dZ}{dt}(\lambda + \delta\lambda) \\ &= 2\lambda^2 C_1 + 2\lambda^2 A_1 + \mathcal{O}(\lambda^3) . \end{aligned} \tag{34}$$

Finally, for the mass term we have $m_0^2 = (m^2 + \delta m^2)/Z$, giving

$$\begin{aligned} \frac{dm^2}{dt} &= -\frac{d}{dt}(\delta m^2) + \frac{1}{Z}\frac{dZ}{dt}(m^2 + \delta m^2) \\ &= -\lambda\, B_1 m^2 + \lambda\, A_1 + \mathcal{O}(\lambda^2) . \end{aligned} \tag{35}$$

The solutions to these equations describe how the parameters evolve as we change μ.

Let us now solve Eq. (34). The right side of this equation depends on the scale μ only implicitly through $\lambda(\mu)$, and is sometimes called the *beta function* for λ: $\beta_\lambda(\lambda)$. Solving,

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - 2(A_1 + C_1)\lambda(\mu_0)\ln(\mu/\mu_0)} . \tag{36}$$

This relates the values of $\lambda(\mu)$ defined at different renormalization scales μ. To interpret the physics of this, let us go back to the one-loop correction to the 4-point function for $\phi\phi \to \phi\phi$ given in Eq. (32). For scattering at momenta p_1^2, the one-loop correction to the amplitude is minimized by choosing $\mu_1^2 \simeq p_1^2$, which gives an amplitude of $\mathcal{M}(p_1^2) \simeq \lambda(\mu_1)$. However, if we try to use the same μ_1 to compute the amplitude for scattering at momentum $p_2^2 \gg p_1^2$, we will get a large logarithm and the higher-order terms will be large. Instead, we should choose $\mu = \mu_2 \sim p_2^2$ to get the amplitude $\mathcal{M}(p_2^2) \simeq \lambda(\mu_2)$. The relation of Eq. (36) tells us how these two couplings are related to each other, and correspondingly how the scattering amplitude evolves with energy (beyond just kinematics).

This relation between couplings defined at different renormalization scales is often called *renormalization group* (RG) evolution. It goes beyond simple perturbation theory in the coupling by resumming potentially large logarithms. This can be seen by expanding the denominator of Eq. (36),

$$\lambda(\mu) \;=\; \lambda(\mu_0)\sum_{n=0}^{\infty}\left[2(A_1+C_1)\lambda(\mu_0)\ln(\mu/\mu_0)\right]^n \;. \tag{37}$$

The expansion parameter is now $\lambda\ln(\mu)$ instead of just λ. By using the renormalization group, we have taken care of the worst of the potentially large logarithms! Furthermore, a pretty good approximation in many cases is to compute quantities using their tree-level expressions, but written with RG-evolved couplings evaluated at the optimal scale. This is sometimes called the *leading-log* approximation. Note that the dependence on μ must cancel out in any physical observable. In the scattering amplitude example discussed here, this will occur once small finite corrections to the field normalization (corresponding mainly to the $\lambda^2 C_2$ term in Eq. (16)) are taken into account.

These ideas can be formalized further in terms of a renormalized *effective action* [8]. Let $i\Gamma^{(n)}(x_1,\ldots,x_n)$ be the renormalized 1PI connected truncated n-point functions of the theory in position space. These are just the Fourier transforms of the objects we have been computing with Feynman diagrams in momentum space. Putting them together, we can form the (1PI) effective action for the theory,[4]

$$\Gamma[\varphi] \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\left[\int\prod_{i=1}^{n}d^4x_i\;\varphi(x_i)\right]\;\Gamma^{(n)}(x_1,\ldots,x_n) \tag{38}$$

$$\simeq\; \int d^4x\;\left[\frac{1}{2}\tilde{Z}(\mu)(\partial\varphi)^2-\frac{1}{2}m^2(\mu)\varphi^2-\frac{\lambda(\mu)}{4!}\varphi^4+(\text{non-local terms})\right]\;, \tag{39}$$

where $\varphi(x)$ is an arbitrary scalar function and $\tilde{Z}(\mu) \simeq 1+\mathcal{O}(\lambda)$. The point of the effective action is that if you compute with it at *tree-level*, you get the full quantum-corrected n-point 1PI truncated Green's functions [8]. The non-local terms in this expression generate the explicitly p^2-dependent parts of the full quantum-corrected n-point functions discussed above. They will be relatively small provided λ is small and we choose $\mu^2 \sim p^2$.

[4]Note that there is also something called a Wilsonian effective action, which is slightly different [1].

3 Symmetries and Anomalies

Symmetries play a central role in all of physics, and they are particularly useful in QFT. There are two specific results concerning symmetries that we have already discussed in this course: *Noether's Theorem* and *Goldstone's Theorem.* Our derivations of these theorems were performed using the classical action, but they can be carried over almost identically to the full quantum theory by replacing the classical action with the full quantum effective action.

When a transformation is a symmetry of the full quantum theory, it can be applied to organize and constrain quantum corrections. In some cases, however, a transformation that is a symmetry of the classical action is not a symmetry of the quantum action. When this occurs, the would-be symmetry is said to be *anomalous.* We discuss and illustrate these features in this section.

3.1 Symmetries and Quantum Corrections

Symmetry considerations are very useful for organizing quantum corrections. In particular, they constrain the types of divergences that are possible. On the other hand, quantum corrections can generate new interactions that were not included in the initial Lagrangian provided they are consistent with the (quantum) symmetries of the theory.

As a first explicit example, consider the real scalar theory with the Lagrangian of Eq. (9). The action is clearly invariant under $\phi_0 \to -\phi_0$, and this implies that any n-point function (or amplitude) with an odd number of ϕ particles must vanish. Suppose we now extend this theory with a new interaction:

$$\begin{aligned} \mathscr{L} &\to \mathscr{L} - \frac{A_0}{3!}\phi_0^3 \\ &= \mathscr{L} - \frac{A}{3!}\phi^3 - \frac{\delta A}{3!}\phi^3 \ , \end{aligned} \tag{40}$$

where $\delta A = (Z^{3/2}A_0 - A)$. The coupling A_0 is a constant with dimensions of mass, and it ruins the invariance under $\phi \to -\phi$. Note, however, that the symmetry would be restored if we also had $A \to -A$. Of course, A is a fixed constant that does not transform, but pretending that it does can be extremely useful. Here, it implies that any n-point function with n odd must be proportional to (an odd number of) A factors. For example, at one-loop we find a correction to the 3-point vertex,

$$\Delta\Gamma^{(3)} \sim \lambda\, A\, (\ln\Lambda + \text{finite}) + \delta A \ . \tag{41}$$

The result is proportional to A, as expected. An important implication of this is that if we start with $A = 0$, quantum corrections will not generate a non-zero value. The new coupling will also correct other parameters in the Lagrangian. Its one-loop contribution to the 2-point function goes like

$$\Delta\Gamma^{(2)} \sim A^2\, (\ln\Lambda + \text{finite}) - \delta m^2 \ . \tag{42}$$

This is a correction to the mass m^2, and it respects the restored symmetry with $A \to -A$.

For a second example, let us expand our scalar theory (with $A = 0$) to include a Dirac fermion ψ [9],

$$\begin{aligned}\mathscr{L} &\to \mathscr{L} + \overline{\psi}_0 i\gamma^\mu \partial_\mu \psi_0 - y_0 \phi_0 \overline{\psi}_0 \psi_0 \\ &= \mathscr{L} + \overline{\psi} i\gamma^\mu \partial_\mu \psi - y\,\phi\,\overline{\psi}\psi - \delta y\,\phi\,\overline{\psi}\psi\ ,\end{aligned} \tag{43}$$

with $\delta y = (Z_\phi^{1/2} Z_\psi y_0 - y)$. The extended theory is symmetric under *chiral transformations*

$$\phi \to -\phi, \quad \psi \to i\gamma^5\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma^5\ . \tag{44}$$

Correspondingly, we have $\overline{\psi}\gamma^\mu\psi \to \overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\psi \to -\overline{\psi}\psi$. This implies that a mass term of the form $M\overline{\psi}\psi$ is forbidden by the symmetry, as is the cubic scalar of Eq. (40).

Now suppose we break this symmetry explicitly by adding a fermion mass term,

$$\begin{aligned}\mathscr{L} &\to \mathscr{L} - M_0\,\overline{\psi}_0\psi_0 \\ &= \mathscr{L} - M\,\overline{\psi}\psi - \delta M\,\overline{\psi}\psi\ ,\end{aligned} \tag{45}$$

where $\Delta M = (Z_\psi M_0 - M)$. While the symmetry discussed above is broken explicitly, it would be restored if the field transformations were accompanied by $M \to -M$ as well. At one-loop, there is a correction to the fermion mass of

$$\Delta\Gamma^{(\bar{\psi}\psi)} \;\sim\; y^2 M\,(\ln\Lambda + \text{finite}) - \delta M\ . \tag{46}$$

This is consistent with the restored symmetry for $M \to -M$. There is also a new divergence corresponding to a ϕ^3 interaction,

$$\Delta\Gamma^{(\phi^3)} \;\sim\; y^3 M\,(\ln\Lambda + \text{finite})\ . \tag{47}$$

To cancel this divergence, we need a ϕ^3 counterterm. This implies that we should have included a ϕ^3 term in the Lagrangian of Eq. (45), along with the fermion mass term, because it is now consistent with the symmetries of the expanded theory. This is an example of quantum corrections "generating" a new type of interaction. In general, all possible (renormalizable) interactions that are consistent with the symmetries of the theory should be included. Note as well that this correction is consistent with the restoration of the symmetry under $A \to -A$ (simultaneously with $M \to -M$).

3.2 Anomalies

A surprising result of quantizing field theories is that quantum corrections sometimes break symmetries of the classical action. When this happens, the symmetry is said to be *anomalous* and the theory is said to have an anomaly [1, 2, 3]. Anomalies are an interesting and important feature of quantum field theories, and it would be easy to spend a whole course discussing them. Due to lack of time, we will only cover a few of the essential aspects of anomalies as they relate to the SM.

In formulating a quantum field theory, one typically needs both an action to define the fields and interactions, and a procedure for regularization and renormalization to deal with apparent divergences. Anomalies can arise when it is impossible to regularize/renormalize the theory in a way to preserves a classical symmetry of the Lagrangian. An equivalent way to think about this is in terms of a path integral formulation of the quantum theory. It turns out that renormalizing the theory is closely related to defining the path integral measure. In this context, anomalous symmetries can be thought of as transformations that leave the action invariant but induce a non-trivial variation in the path integral measure.

To identify an anomaly in a continous symmetry, it sufficient to find a non-vanishing operator matrix element involving the divergence of the corresponding Noether current:

$$\langle \mathcal{O}\, \partial_\mu j^\mu \rangle \neq 0, \tag{48}$$

where $\mathcal{O}$ is any operator in the theory. We will use this approach below.

Chiral fermions can be a source of anomalies in four dimensions. Consider a theory with left-handed fermions ψ_{L_i} and right-handed fermions ψ_{R_j}, an Abelian global symmetry G, and an Abelian gauge symmetry H. Let us assume the LH fermions have charges $Q^G_{L_i}$ and $Q^H_{L_i}$ under these groups and the RH fermions have charges $Q^G_{R_j}$ and $Q^H_{R_j}$. The Noether current for the global symmetry G is

$$j^G_\mu = \sum_i Q^G_{L_i} \bar{\psi}_{L_i} \gamma^\mu \psi_{L_i} + \sum_j Q^G_{R_j} \bar{\psi}_{R_j} \gamma^\mu \psi_{R_j}. \tag{49}$$

One can show that no matter how one regularizes the theory (in a Lorentz-invariant way), diagrams of the form shown in the left panel of Fig. 3 lead to a non-zero matrix element of the divergence of the G current with a pair H gauge bosons with coefficient

$$\langle A_\mu A_\nu\, \partial^\lambda j^G_\lambda \rangle \;\propto\; \sum_i (Q^H_{L_i})^2 Q^G_{L_i} - \sum_j (Q^H_{R_j})^2 Q^G_{R_j}\ . \tag{50}$$

Unless this combination of charges vanishes, the expectation value is non-zero and the global symmetry G is anomalous. Note that the anomaly vanishes automatically if the theory is non-chiral, with all the LH and RH fermions coming in pairs with equal charges.[5] It is straightforward to generalize this result to non-Abelian symmetries, and we will do so below.

An anomaly in a global symmetry leads to interesting physical effects in the theory. However, an anomaly in a gauge "symmetry" would be disastrous since it would lead to a distinction between field configurations that are supposed to be physically equivalent. Therefore an important consistency condition for gauge theories with chiral fermions is that the gauge symmetries (treated as classical global symmetries) be anomaly-free. A sufficient condition for this to occur is that the sum of all fermion-loop triangle diagrams with three external gauge boson legs vanish – see the right panel of Fig. 3. These diagrams are proportional to anomaly coefficients which depend on the chiral fermion representations.

[5] If we had written all the SM fermion reps in terms of 2-component LH spinors, there would be no pesky relative minus sign in Eq. (50).

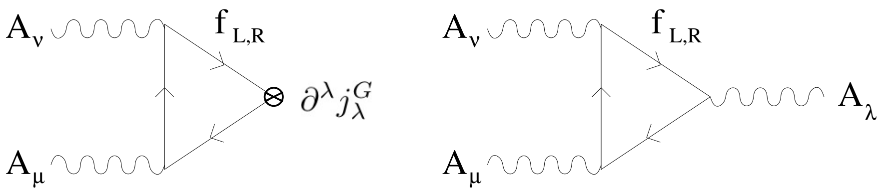

Figure 3: Fermion triangle diagrams contributing to global (left) and gauge (right) anomalies.

For the SM, the non-trivially vanishing anomaly coefficients are:

$$SU(3)^3_c \quad \propto \quad \sum_L tr(t^a_c\{t^b_c, t^c_c\}) - (L \to R) \tag{51}$$

$$SU(3)^2_c \times U(1)_Y \quad \propto \quad \sum_L tr(t^a_c t^b_c Y) - (L \to R) \tag{52}$$

$$SU(2)^3_L \quad \propto \quad \sum_L tr_L(t^p_L\{t^q_L, t^r_L\}) - (L \to R) \tag{53}$$

$$SU(2)^2_L \times U(1)_Y \quad \propto \quad \sum_L tr_L(t^p_L t^q_L Y) - (L \to R) \tag{54}$$

$$U(1)^3_Y \quad \propto \quad \sum_L Y^3 - (L \to R) \tag{55}$$

$$(grav)^2 U(1)_Y \quad \propto \quad \sum_L Y - (L \to R) \tag{56}$$

Here, the sum $\sum_L$ runs over all left-handed fermion reps, and similarly for R. Sometimes you will see $\sum_L(...) = tr_L(...)$. These anomaly coefficients are just the group theoretic factors associated with the corresponding triangle loops weighted by a relative factor of minus one for chirality. Note that mixed anomalies with a single non-Abelian factor, like $SU(3)^2_c SU(2)_L$ or $SU(2)_L U(1)^2_Y$, vanish automatically since they all involve the trace of a single non-Abelian generator. The last condition is only needed if we want to eventually couple the theory in a consistent way to gravity (which we do).

As an explicit example, consider the $SU(3)^2_c \times U(1)_Y$ anomaly coefficient in the SM. It is

$$A_{331} = tr_L(t^a_c t^b_c Y) - (L \to R) \ . \tag{57}$$

The L part of the trace gets contributions from Q. For this, there are two **3** reps of $SU(3)_c$, one each for the two $SU(2)_L$ components u_L and d_L, and both have hypercharge $Y = 1/6$. The R part of the trace comes from u_R and d_R which are both **3** reps of $SU(3)_c$ and have hypercharges $Y = 2/3$ and $-1/3$. Putting things together, and using $tr(t^a_3 t^b_3) = \delta^{ab}/2$, we find

$$A_{331} = n_g \left[\left(\frac{1}{2} \times 2 \times \frac{1}{6} \right) \quad - \quad \left(\frac{1}{2} \times \frac{2}{3} \quad - \quad \frac{1}{2} \times \frac{1}{3} \right) \right] = 0 \ , \tag{58}$$

where $n_g = 3$ is the number of generations.

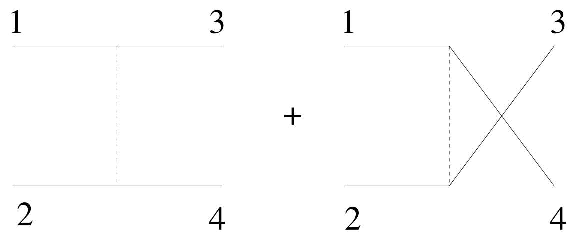

Figure 4: Contributions to $\psi\psi \to \psi\psi$ scattering mediated by the heavy scalar ϕ.

4 Effective Field Theories

An *effective field theory* (EFT) is a field theory that describes the low-energy dynamics of a more complicated theory in terms of only the light degrees of freedom [5, 6, 9, 10]. By their construction, EFTs are usually only useful at energies well below a built-in UV cutoff. Despite this limitation, EFTs are very often the most convenient way to calculate experimental observables at low energies. Indeed, the modern view of the SM is that it is the EFT limit of a more complicated theory [11].

To illustrate how EFTs work, let us begin with a simple and familiar example:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \overline{\psi}i\gamma^\mu\partial_\mu\psi - y\,\phi\overline{\psi}\psi \ . \tag{59}$$

This theory is renormalizable, and it has a symmetry under $\phi \to -\phi$ and $\psi \to i\gamma^5\psi$ that forbids odd terms in ϕ and a mass for ψ. Suppose we are interested in studying the theory at energies that are far below the mass m of the scalar ϕ. At such energies, it is impossible to produce ϕ particles directly. However, they can still contribute to interactions among ψ particles. We would like to formulate an EFT containing only the massless ψ particles that takes this into account.

The dominant effect of ϕ on the interactions among ψ particles can be seen in the leading contributions to the scattering process $\psi\psi \to \psi\psi$ shown in Fig. 4. In the diagrams contributing to the amplitude, the massive ϕ particle only appears in the intermediate propagator. The contribution to the amplitude from the first diagram is

$$\begin{aligned} -i\mathcal{M}_1 &= (-iy)^2(\bar{u}_3u_1)(\bar{u}_4u_2)\,\frac{i}{t-m^2} && (60)\\ &= i\frac{y^2}{m^2}\,(\bar{u}_3u_1)\,(\bar{u}_4u_2)\left(1+\frac{t}{m^2}+\frac{t^2}{m^4}+\dots\right)\ , && (61)\end{aligned}$$

where p_1 and p_2 are the incoming momenta, p_3 and p_4 are the outgoing momenta, $t = (p_1 - p_3)^2$, and we have expanded the propagator in powers of t/m^2 in the second line. A similar expression can be derived for the second diagram with an expansion in u/m^2, with $u = (p_1 - p_4)^2$.

To formulate an EFT for the low-energy limit of the full theory defined by Eq. (59), we must construct an effective Lagrangian involving only the light ψ fields that reproduces the predictions of the full theory for $E \ll m$. This procedure is called *matching*, and we say that we have *integrated out* the scalar to obtain the EFT. All the terms in Eq. (61) can be reproduced with operators involving only ψ fields, although an infinite number of operators is needed to do so. Fortunately, we only have to reproduce a finite number of them for $|t| \ll m^2$ provided we only want to compute to a finite accuracy. The dominant first term in Eq. (61) can be obtained with the effective operator

$$-\mathscr{L}_{EFT} \supset -\frac{y^2}{2m^2}\,(\overline{\psi}\psi)(\overline{\psi}\psi)\ . \tag{62}$$

The other terms in Eq. (61) can be generated by including similar four-fermion operators with derivatives acting on the fields and more powers of m^2 in the denominator.

In the SM, it is standard practice to formulate the low-energy limit of the weak interactions in terms of an EFT with no explicit $W^\pm$ or Z^0 vector bosons. For example, consider the decay of a muon by way of a W^- to $\nu_\mu\, e\bar{\nu}_e$. The amplitude is

$$\begin{aligned} -i\mathcal{M} &= \bar{u}_e(-i\frac{g}{\sqrt{2}}\gamma^\mu P_L)v_{\bar{\nu}_e}\;\bar{u}_{\nu_\mu}(-i\frac{g}{\sqrt{2}}\gamma^\nu P_L)u_\mu\;\frac{i}{p^2-m_W^2}\left(-\eta_{\mu\nu}+p_\mu p_\nu/m_W^2\right) \qquad (63)\\ &= -i\frac{g^2}{2m_W^2}\left(\frac{1}{1-p^2/m_W^2}\right)(\eta_{\mu\nu}-p_\mu p_\nu/m_W^2)\,(\bar{u}_e\gamma^\mu P_L v_{\bar{\nu}_e})\left(\bar{u}_{\nu_\mu}\gamma^\nu P_L u_\mu\right)\ . \end{aligned}$$

The momenta here are all less than the mass of the muon, which is a lot smaller than the mass of the W. Thus, we can expand this amplitude in powers of p^2/m_W^2, and it is a good approximation (up to corrections of size p^2/m_W^2) to keep only the leading term. At this order, the amplitude becomes

$$-i\mathcal{M} \;\simeq\; i\frac{g^2}{2m_W^2}\,(\bar{u}_e\gamma^\mu P_L v_{\bar{\nu}_e})\left(\bar{u}_{\nu_\mu}\gamma_\mu P_L u_\mu\right)\ . \tag{64}$$

Exactly the same amplitude could have been obtained if we had started from a Lagrangian containing the interaction

$$\mathscr{L} \;\supset\; \frac{4G_F}{\sqrt{2}}\,(\bar{e}\gamma_\mu P_L\nu_e)\,(\bar{\nu}_\mu\gamma^\mu P_L\mu)\ , \tag{65}$$

where $G_F = g^2/8m_W^2 = 1/(2\sqrt{2}v^2)$ is called the *Fermi constant*. Higher-order terms in the expansion could also be reproduced by including operators with derivatives acting on the fields. More generally, the leading effects of the weak interaction at low energies can be formulated in terms of a set of four-fermion effective operators suppressed by g^2/m_W^2 and g^2/m_Z^2. In fact, this is how the theory of the weak force was first developed by Fermi and others, well before non-Abelian gauge theories and the Higgs mechanism were understood.

A central feature of both EFTs derived above are they they are non-renormalizable, even though the underlying theories (the Yukawa model and the SM) are. This is not a problem, and it provides further insight into how to think about non-renormalizable theories. Recall that we argued that non-renormalizable theories can be used to make testable predictions provided they are only applied at energies much smaller than the mass scale m_{EFT} that suppresses the higher-dimensional operators. Even though quantum corrections in such theories generate an infinite number of new types of divergences, they correspond to operators of increasing higher dimension suppressed by more powers of m that are numerically small. These are precisely the same types of operators we neglected in the leading expressions for the EFTs derived here. In general, the modern interpretation is that non-renormalizable QFTs are EFTs valid for momenta $p^2 \ll m_{EFT}^2$. As p^2 grows to near m_{EFT}^2, the EFT treatment breaks down because the full dynamics of the heavy physics characterized by m_{EFT} must be taken into account. On the other hand, as p^2 becomes very small compared to m_{EFT}^2 the non-renormalizable operators become less and less important and the theory approaches a renormalizable theory.

In the examples considered above, we were also lucky enough to know the full, weakly-interacting (by assumption) high-energy theories. Things are not always so simple in practice. Often, we do not know the high-energy theory because we are only able to probe the light degrees of freedom experimentally. In other cases, we do know the high-energy theory, but it is strongly coupled near the matching scale. A well-known example of this is QCD, with the high-energy theory consisting of quarks and gluons and the low-energy theory built from nucleons and mesons. In this case, the matching is more extreme, and the low- and high-energy degrees of freedom are completely different.

These general features of EFTs have shaped the modern interpretation of the SM as the low-energy limit of a more complicated theory that describes our Universe. If there exists new physics beyond the SM, it is likely to show up in the form of non-renormalizable operators built from SM fields. Despite many experimental searches, no definitive evidence of such operators has been found. Even so, a potential example of this are the observed neutrino masses [12]. These are forbidden in the SM at the renormalizable level, but they can be induced by an operator of the form [9, 11]

$$-\mathscr{L} \supset \frac{1}{M_N^2}(HL)^2 \ . \tag{66}$$

The observed neutrino mixings suggest $M_N \sim 10^{13}$ GeV. Another higher-dimensional operator that has been searched for extensively is [9]

$$-\mathscr{L} \supset \frac{1}{M^2}\,QQQL \ . \tag{67}$$

This operator can induce proton decays such as $p \to \pi^0\,\bar{e}^+$. Despite extensive searches, proton decay has never been observed and a lifetime of $\tau_p \gtrsim 10^{32}\,$yr has been set, with the specific limit depending on the decay mode [13]. This translates into $M \gtrsim 10^{16}$ GeV.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[2] M. Srednicki, "Quantum field theory," *Cambridge, UK: Cambridge U. Press (2006) 660 p*
http://web.physics.ucsb.edu/ mark/qft.html

[3] M. D. Schwartz, "Quantum Field Theory and the Standard Model," *Cambridge, UK: Cambridge U. Press (2013) 863 p*

[4] J. C. Collins, "Renormalization: General theory," [hep-th/0602121]; J. C. Collins, "The Problem of scales: Renormalization and all that," [hep-ph/9510276]; J. C. Collins, "Renormalization. An Introduction To Renormalization, The Renormalization Group, And The Operator Product Expansion," Cambridge, Uk: Univ. Pr. (1984) 380p.

[5] H. Georgi, "Weak Interactions and Modern Particle Theory," Menlo Park, Usa: Benjamin/cummings (1984) 165p http://www.people.fas.harvard.edu/ hgeorgi/weak.pdf .

[6] T. Cohen, "As Scales Become Separated: Lectures on Effective Field Theory," PoS **TASI2018**, 011 (2019) [arXiv:1903.03622 [hep-ph]].

[7] A. J. Buras, "Weak Hamiltonian, CP violation and rare decays," hep-ph/9806471.

[8] See, for example, R. H. Brandenberger, "Quantum Field Theory Methods and Inflationary Universe Models," Rev. Mod. Phys. **57**, 1 (1985).

[9] M. A. Luty, "2004 TASI lectures on supersymmetry breaking," [hep-th/0509029].

[10] D. B. Kaplan, "Five lectures on effective field theory," [nucl-th/0510023].

[11] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[12] A. Strumia and F. Vissani, "Neutrino masses and mixings and...," hep-ph/0606054.

[13] V. Takhistov [Super-Kamiokande Collaboration], "Review of Nucleon Decay Searches at Super-Kamiokande," arXiv:1605.03235 [hep-ex].