P528 Notes #6: The Standard Model

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We now have all the pieces we need to write down the Lagrangian for the Standard Model! To start, we will discuss chiral fermions, which feature prominently. Next, we will present the general structure of the SM, with a focus on gauge invariance and electroweak symmetry breaking, followed by a discussion of flavour, and then an overview of the Feynman rules.

1 Warmup: Chiral Fermions

We have been working with scalar, fermion, and vector fields. They correspond to particles with definite spins. They also form representations of the Lorentz group, which is generated by the set of boosts and rotations on spacetime. Under this group, the fields transform as

$$\begin{aligned} x^\mu &\to x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ \phi(x) &\to \phi'(x') = \phi(x) \\ \psi_a(x) &\to \psi'(x') = [M(\Lambda)]_{ab}\psi_b(x) \\ A^\mu &\to A'^\mu = \Lambda^\mu{}_\nu A^\nu(x)\ . \end{aligned} \tag{1}$$

Recall that the indices on the fermions are 4-component Dirac indices.

Just like the gauge groups we discussed previously, the Lorentz group is a Lie group. Group elements can be identified by six coordinates, and the underlying Lie algebra has six generators [1]. These generators correspond to three rotation generators (about the x, y, z axes) and three boost generators (along the x, y, z axes). To figure out the possible representations of the Lorentz group, it is just a matter of finding matrices that satisfy the Lie algebra relations of the generators.

In the case of the 4-component *Dirac* fermions we have been using, the general form of the transformation matrices in the chiral basis is [2]

$$M(\Lambda) = \left(\begin{array}{cc} e^{-i\alpha^i\sigma^i/2} & 0 \\ 0 & e^{-i\alpha^{i*}\sigma^i/2} \end{array}\right)\ , \tag{2}$$

where $\alpha^i = (\theta^i + i\beta^i)$, $i = 1, 2, 3$, are the group coordinates corresponding to the Lorentz transformation matrix Λ with rotation angles θ^i and boosts β^i. In contrast to the Lie groups we studied previously, the group coordinates are now complex and the corresponding representation matrices are no longer unitary in general. The reason for this is that the Lorentz group is *non-compact* because the boost coordinates span an infinite range.

The key feature of the transformation matrix of Eq. (2) is that it has a 2×2 block diagonal form. Writing a 4-component Dirac fermion as two 2-component pieces,

$$\psi = \left(\begin{array}{c} \chi \\ \bar{\xi} \end{array}\right)\ , \tag{3}$$

this implies that χ and $\bar{\xi}$ transform independently under Lorentz.[1] Thus, 4-component Dirac fermions are a *reducible representation* of the Lorentz group. The 2-component fermions χ and $\bar{\xi}$ said to transform in the left- and right-handed 2-component *Weyl* representations of the Lorentz group.

On account of this reducibility, it is possible to write consistent Lorentz-invariant theories for 2-component fermions on their own, without any reference to Dirac fermions. For example, the minimal Lagrangian for a left-handed 2-component fermion is

$$\mathscr{L} \;=\; \overline{\psi}_L i\gamma^\mu\partial_\mu\psi_L \;=\; \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi \;, \tag{4}$$

where

$$\psi_L := \left(\begin{array}{c} \chi \\ 0 \end{array}\right) \;. \tag{5}$$

Interpreted as a quantum theory, this Lagrangian describes a massless fermion with left-handed helicity (spin antiparallel to the direction of motion) and its antiparticle with right-handed helicity (spin parallel to motion). Similarly, the minimal Lagrangian for a right-handed 2-component fermion is

$$\mathscr{L} \;=\; \overline{\psi}_R i\gamma^\mu\partial_\mu\psi_R \;=\; \bar{\xi}^\dagger i\sigma^\mu\partial_\mu\bar{\xi} \;, \tag{6}$$

where

$$\psi_R := \left(\begin{array}{c} 0 \\ \bar{\xi} \end{array}\right) \;. \tag{7}$$

The left- and right-handed 2-component fermion representations are essentially complex conjugates of each other. More precisely, given a left-handed 2-component fermion χ, we can make a right-handed 2-component fermion $\bar{\chi}$ by

$$\bar{\chi} := i\sigma^2(\chi)^* \;. \tag{8}$$

Similarly, given the right-handed 2-component fermion $\bar{\xi}$, the corresponding left-handed fermion is

$$\xi := -i\sigma^2(\bar{\xi})^* \;. \tag{9}$$

The relative sign here is chosen such that applying the conjugation operation of Eq. (8) to ξ, we get back $\bar{\xi}$ (and vice-versa).

At this point, you might be wondering why to use Dirac fermions at all. To see the reason, recall the basic Lagrangian for a massive Dirac fermion with components $\psi = (\chi, \bar{\xi})^t$:

$$\begin{aligned} \mathscr{L} &= \overline{\psi} i\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \\ &= \overline{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \overline{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right) \\ &= \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \bar{\xi}^\dagger i\sigma^\mu\partial_\mu\bar{\xi} - m(\bar{\xi}^\dagger\chi + \chi^\dagger\bar{\xi}) \;. \end{aligned} \tag{10}$$

Thus, our basic Dirac mass term mixes the left- and right-handed components. Combining them into a single Dirac fermion is a convenient way to avoid diagonalizing mass matrices.

[1] Note that the bar on $\bar{\xi}$ here is part its name, and does not refer to any sort of conjugation (yet).

2 Structure of the Standard Model

We now have all the pieces needed to assemble the Standard Model (SM) [3, 4, 5, 6]. This theory provides an excellent description of the strong, weak, and electromagnetic forces, and its predictions agree with a very broad range of measurements. Gravity is not described by the SM since this force is exceedingly weak and almost always neglible in particle physics experiments. However, for energies well below $E \ll 10^{18}$ GeV and small densities, it can be added to the SM as a quantum effective field theory using the same basic tools we have covered in the course.

2.1 Group Structure

The structure of the SM is based on gauge invariance under the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Of these factors, $SU(3)_c$ corresponds to the strong force, while $SU(2)_L \times U(1)_Y$ combine to produce the weak and electromagnetic forces. Having fixed the underlying gauge group, all we need to do is specify the matter content and the vacuum structure.

The fermionic matter content comes in three identical copies called *families*. Each family consists of the following representations under $SU(3)_c \times SU(2)_L \times U(1)_Y$:

$$\begin{aligned}
Q_L &= (\mathbf{3}, \mathbf{2}, 1/6) = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
u_R &= (\mathbf{3}, \mathbf{1}, 2/3) \\
d_R &= (\mathbf{3}, \mathbf{1}, -1/3) \\
\\
L_L &= (\mathbf{1}, \mathbf{2}, -1/2) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
e_R &= (\mathbf{1}, \mathbf{1}, -1)
\end{aligned} \tag{11}$$

Each of these is a 2-component fermion written in 4-component notation. Note that these representations do not come in balanced LR and RH pairs; instead the LH and RH quark and lepton fields have different gauge charges. For Q_L and L_L, we have written out the $SU(2)_L$ components explicitly (which you should not confuse with Dirac components). The Q_L, u_R, and d_R fields transform non-trivially under $SU(3)_c$ and are called *quarks*, while the $SU(3)_c$-neutral L_L and e_R fields are called *leptons*. Each quark has three colour components, which we have not written out explicitly.

In addition to three families of fermions, there is a single complex scalar Higgs field

$$H = (\mathbf{1}, \mathbf{2}, 1/2) = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} . \tag{12}$$

We will write the gauge fields for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ factors as

$$\begin{aligned}
G^a_\mu &\sim (\mathbf{8}, \mathbf{1}, 0) \\
W^p_\mu &\sim (\mathbf{1}, \mathbf{3}, 0) \\
B_\mu &\sim (\mathbf{1}, \mathbf{1}, 0)
\end{aligned} \tag{13}$$

Recall that the **8** of $SU(3)_c$ is the adjoint, as is the **3** of $SU(2)_L$.

Under $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformations, a field ψ transforms according to

$$\begin{aligned}
\psi_{ir} \to \psi'_{ir} &:= U^{(3)}_{ij} U^{(2)}_{rs} U^{(1)} \, \psi_{js} \qquad (14) \\
&= (e^{i\alpha^a t^a_c})_{ij} \, (e^{i\beta^p t^p_L})_{rs} \, (e^{i\gamma Y}) \, \psi_{js} \\
&= [\delta_{ij}\delta_{rs} + i\alpha^a (t^a_c)_{ij}\delta_{rs} + i\delta_{ij}\beta^p (t^p_L)_{rs} + i\delta_{ij}\delta_{rs}\gamma Y + \ldots] \, \psi_{js} \ . \qquad (15)
\end{aligned}$$

That is, ψ carries $SU(3)_c$ (i and j) and $SU(2)_L$ (r and s) indices, and each of these product subgroups acts independently relative to these indices. The quantities α^a, β^p, and γ are the group transformation parameters that apply universally to all representations. When a field transforms as a singlet under $SU(3)_c$ or $SU(2)_L$, the corresponding representation generators vanish and we do not need to include an index for that group on the field. Explicitly,

$$Q_L = (Q_L)_{ir}, \quad u_R = (u_R)_i, \quad d_R = (d_R)_i \ , \quad L_L = (L_L)_r, \quad e_R = (e_R) \ . \qquad (16)$$

Woohoo!

2.2 Lagrangian

The SM Lagrangian takes the form

$$\mathscr{L} = \mathscr{L}_{gauge} + \mathscr{L}_{Higgs} + \mathscr{L}_{Yukawa} \ . \qquad (17)$$

The gauge piece is completely fixed by gauge invariance:

$$\begin{aligned}
\mathscr{L}_{gauge} &= -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^p_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 \\
&\quad + \overline{Q}_L i\gamma^\mu D_\mu Q_L + \overline{u}_R i\gamma^\mu D_\mu u_R + \overline{d}_R i\gamma^\mu D_\mu d_R \qquad (18) \\
&\quad + \overline{L}_L i\gamma^\mu D_\mu L_L + \overline{e}_R i\gamma^\mu D_\mu e_R \ ,
\end{aligned}$$

where each covariant derivative takes the form

$$D_\mu = \partial_\mu + ig_s \, t^a_c G^a_\mu + ig \, t^p_L W^p_\mu + ig' \, Y B_\mu, \qquad (19)$$

with t^a_c the appropriate $SU(3)_c$ generators for the coresponding rep ($t^a_c = 0$ for the trivial rep), t^p_L the generators for $SU(2)_L$ ($t^p_L = 0$ for the trivial rep), and Y is the charge of the field under $U(1)_Y$ and is called *hypercharge*. The Higgs part is

$$\mathscr{L}_{Higgs} = \left| \left(\partial_\mu + ig \, \frac{\sigma^p}{2} W^p_\mu + ig' \frac{1}{2} B_\mu \right) \, H \right|^2 - \left(-\mu^2 |H|^2 + \frac{\lambda}{2}|H|^4 \right) \ . \qquad (20)$$

This potential induces spontaneous symmetry breaking. Finally, the third set of terms in the SM Lagrangian correspond to scalar-fermion *Yukawa* interactions,

$$\mathscr{L}_{Yukawa} = -y_u \overline{Q}_L \tilde{H} u_R - y_d \overline{Q}_L H d_R - y_e \overline{L}_L H e_R + (h.c.) \ , \qquad (21)$$

where $\tilde{H} := i\sigma^2 \Phi = (H^{0*}, \, -H^{+*})^t$. These interactions are the most general ones we can write (at the renormalizable level) that are consistent with gauge invariance given the charges of Eq. (11). Note that the gauge charges forbid fermion mass terms at this stage.

2.3 Electroweak Symmetry Breaking and Masses

The first step in working out the implications of this Lagrangian is to determine the vacuum structure. The Higgs potential leads to spontaneous symmetry breaking. To study this, it is conveninent to choose a gauge (called the *unitarity gauge*) such that

$$H(x) = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix} , \tag{22}$$

where $v = \sqrt{\mu^2/\lambda}$, and the real scalar h field is called the Higgs boson. This expectation value has important consequences for the rest of the theory. From the Higgs kinetic term we obtain masses for some of the W^p_μ and B^μ gauge bosons. Inserting this form for the Higgs field into Eq. (21), we also obtain masses for the fermions.

Symmetry breaking in the SM has the same form as the $SU(2) \times U(1)$-invariant theory we considered previously. Applying an arbitrary $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformation to the vacuum state chosen above, we see that this vacuum is invariant under $SU(3)_c$ as well as the Abelian subgroup of $SU(2)_L \times U(1)_Y$ with generator

$$Q := t^3_L + Y \; . \tag{23}$$

We identify this maximal unbroken electroweak subgroup with the $U(1)_{em}$ invariance of electromagnetism. The Q generator defined here corresponds to electric charge, and we expect a massless vector boson to go with it.

To verify this, we construct the gauge boson mass matrix generated by the covariant kinetic term for the Higgs field. This leads to

$$|D_\mu H|^2 \to \frac{1}{2}(\partial h)^2 + \frac{1}{2}\frac{v^2}{2}\left[g^2[(W^1_\mu)^2 + (W^2_\mu)^2] + (-gW^3_\mu + g'B_\mu)^2\right] + \ldots \tag{24}$$

From this expression it is clear that two orthogonal linear combinations of W^1_μ and W^2_μ obtain equal masses. It is convenient to choose the two combinations that have definite charges $Q = \pm 1$ under electromagnetism,

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right) \; . \tag{25}$$

We identify them with the $W^\pm$ vector bosons of the weak interaction, and they have mass

$$m^2_W = \frac{g^2}{2}v^2 \; . \tag{26}$$

For W^3_μ and B_μ we get a squared mass matrix of

$$M^2 = \frac{v^2}{2}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \; . \tag{27}$$

As expected, this matrix has a zero eigenvalue corresponding to the photon A_μ. The other linear combination of W^3_μ and B_μ is called the Z^0 vector boson. These mass eigenstates are related to the fields in the original basis by

$$\left(\begin{array}{c} Z_\mu \\ A_\mu \end{array}\right) = \left(\begin{array}{cc} c_W & -s_W \\ s_W & c_W \end{array}\right)\left(\begin{array}{c} W^3_\mu \\ B_\mu \end{array}\right) , \tag{28}$$

where the *weak mixing angle* θ_W is defined by

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} , \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} . \tag{29}$$

While the photon is massless, the Z^0 vector boson has mass

$$m_Z^2 = \left(\frac{g^2+g'^2}{2}\right) v^2 . \tag{30}$$

The longitudinal components of the massive $W^\pm$ and Z^0 vectors account for the missing NGBs from the three broken electroweak generators. Since the new mass eigenstate vector fields we have defined above are related to the original gauge eigenstates by orthogonal transformations, the kinetic terms for the mass eigenstate vectors are also canonical.

Interactions between these electroweak vector boson mass eigenstates and the fermions of the SM are dictated by the gauge-covariant derivatives. These take the form (with $t^p = 0$ for $SU(2)_L$ singlets)

$$\begin{aligned} D_\mu &\supset igt^pW^p_\mu + ig'YB_\mu \\ &= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] \\ &\qquad + i(gc_Wt^3 - s_Wg'Y)Z_\mu + i(gs_Wt^3 + g'c_WY)A_\mu \\ &= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] + i\bar{g}(t^3 - s_W^2Q)\,Z_\mu + ie\,Q\,A_\mu . \end{aligned} \tag{31}$$

In the last line, we have implicitly defined the couplings

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = gs_W = g'c_W , \qquad \bar{g} = \sqrt{g^2+g'^2} . \tag{32}$$

While the SM has many individual vector boson interaction terms, we see that they are all essentially fixed by the values of g, g', and v from the underlying gauge-invariant theory. Measurements of the electroweak sector of the SM find that [7]

$$\begin{aligned} m_W &\simeq 80.4\text{ GeV}, \quad m_Z \simeq 91.2\text{ GeV}, \quad v \simeq 174\text{ GeV}, \\ s_W^2 &\simeq 0.23, \quad g \simeq 0.65, \quad g' \simeq 0.35, \quad e^2/4\pi \simeq 1/137. \end{aligned} \tag{33}$$

Note that not all the values of these measurable masses and couplings are independent in the underlying theory. We will see that this allows for very stringent experimental tests of the SM.

The remaining parts of the SM Lagrangian to deal with are the Yukawa terms. Rewriting the Higgs scalar doublet in terms the new vacuum-friendly field variables, the Yukawa interactions become

$$\begin{aligned} -\mathscr{L}_{Yukawa} &= y_u \overline{Q}_L \tilde{H}\, u_R + y_d \overline{Q}_L H\, d_R + y_e \overline{L}_L H\, e_R + (h.c.) \\ &= y_u\,(v + h/\sqrt{2})\,\overline{u}_L u_R + y_d\,(v + h/\sqrt{2})\,\overline{d}_L d_R + y_e\,(v + h/\sqrt{2})\,\overline{e}_L e_R + (h.c.)\ . \end{aligned} \tag{34}$$

This expression consists of Dirac mass terms for the fermions together with fermion-Higgs boson interactions:

$$m_i = y_i\, v\ . \tag{35}$$

In other words, the mass of each SM fermion is proportional to how strongly it couples to the Higgs field.

3 Flavour in the Standard Model

In our discussion above, we did not say anything about the three families of quarks and leptons to simplify the discussion. There is a very interesting story here, and we turn to it now. The SM contains three copies of all the fermion representations that we call *families*, *generations*, or *flavours*. They are (in order of increasing mass)

$$\text{Quarks}: \quad \left\{ \begin{array}{llll} u_{L,R} & c_{L.R} & t_{L,R} & Q = +2/3 \\ d_{L,R} & s_{L,R} & b_{L,R} & Q = -1/3 \end{array} \right. \tag{36}$$

$$\text{Leptons}: \quad \left\{ \begin{array}{llll} \nu_{e_L} & \nu_{\mu_L} & \nu_{\tau_L} & Q = \ \ 0 \\ e_{L,R} & \mu_{L,R} & \tau_{L,R} & Q + -1 \end{array} \right. \tag{37}$$

The first column corresponds to the first generation, the second column to the second generation, and the third column to the third. The elements of each generation have identical sets of $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum numbers (*i.e.* representations) but differ greatly in their masses.

Instead of writing out all three generations explicitly, it is much easier to use a condensed notation with a generation index $A = 1, 2, 3$. For example, we will write u_{R_A} with

$$u_{R_{A=1}} = u_R\ , \quad u_{R_2} = c_R\ , \quad u_{R_3} = t_R\ , \tag{38}$$

and similarly for the other states. Since all three generations have identical quantum numbers, we can choose our field variables such that all the gauge-covariantized kinetic terms are diagonal in generation space and canonically normalized. That is

$$\mathscr{L}_{gauge} \supset \overline{Q}_{L_A} i\gamma^\mu D_\mu Q_{L_A} + \overline{u}_{R_A} i\gamma^\mu D_\mu u_{R_A} + \ldots \tag{39}$$

This choice of field variables is sometimes called the *gauge eigenbasis*. We will always implicitly start off with this basis and work from there.

Going back to the Yukawa interactions, gauge invariance allows them to have a non-trivial family-mixing structure. The most general set of gauge-invariant Yukawa terms is

$$\begin{aligned}
-\mathscr{L}_{Yukawa} &= y_{u_{AB}}\overline{Q}_{L_A}\tilde{H}\,u_{R_B} + y_{d_{AB}}\overline{Q}_{L_A}H\,d_{R_B} + y_{e_{AB}}\overline{L}_{L_A}H\,e_{R_B} + (h.c.) \\
&= (v+h/\sqrt{2})\overline{u}_{L_A}y_{u_{AB}}u_{R_B} + (v+h/\sqrt{2})\overline{d}_{L_A}y_{d_{AB}}d_{R_B} + (v+h/\sqrt{2})\overline{e}_{L_A}y_{e_{AB}}e_{R_B} + (h.c.) \\
&= (v+h/\sqrt{2})\,\overline{u}_L y_u u_R + (v+h/\sqrt{2})\,\overline{d}_L y_d d_R + (v+h/\sqrt{2})\,\overline{e}_L y_e e_R + (h.c.)
\end{aligned} \tag{40}$$

In the third line, we have implicitly contracted the generation indices to write this expression in terms of matrices and row and column vectors in generation space.

The portions of the expression above involving v are mass matrices for the up- and down-type quarks and the charged leptons. Their most general form is a set of (non-diagonal) 3×3 complex matrices. To do perturbation theory, we should diagonalize them. Any complex matrix can be diagonalized by a pair of unitary matrices. To achieve this, define a new set of fields according to

$$\begin{aligned}
u_{L_A} &= V^L_{u_{AB}}\,u'_{L_B}\ , \qquad u_{R_A} = V^R_{u_{AB}}\,u'_{R_B}\ , \\
d_{L_A} &= V^L_{d_{AB}}\,d'_{L_B}\ , \qquad d_{R_A} = V^R_{d_{AB}}\,d'_{R_B}\ , \\
\\
e_{L_A} &= V^L_{e_{AB}}\,e'_{L_B}\ , \qquad e_{R_A} = V^R_{e_{AB}}\,e'_{R_B}\ , \\
\nu_{L_A} &= V^L_{\nu_{AB}}\,\nu'_{L_B}\ ,
\end{aligned} \tag{41}$$

where the $V_f^{L,R}$ are unitary matrices in generation space. We can choose them such that they bi-diagonalize the Yukawa interaction matrices. That is

$$\begin{aligned}
V_u^{L\dagger}y_u V_u^R &= \frac{1}{v}\,diag(m_u, m_c, m_t) \\
V_d^{L\dagger}y_d V_d^R &= \frac{1}{v}\,diag(m_d, m_s, m_b) \\
V_e^{L\dagger}y_e V_e^R &= \frac{1}{v}\,diag(m_e, m_\mu, m_\tau)
\end{aligned} \tag{42}$$

In terms of the primed fields, the Yukawa interactions containing the mass terms are now diagonal. For example

$$\begin{aligned}
-\mathscr{L}_{Yukawa} &\supset (v+h/\sqrt{2})\,u_L y_u u_R \\
&= (v+h/\sqrt{2})\,\overline{u}'_L\,(V_u^{L^\dagger}y_u V_u^R)\,u'_R \\
&= (1+h/\sqrt{2}v)\,\Big(m_u\,\overline{u}'_L u'_R + m_c\,\overline{c}'_L c'_R + m_t\,\overline{t}'_L t'_R\Big)\ .
\end{aligned} \tag{43}$$

Since these field transformations are unitary (and global), the fermion kinetic terms retain their generation-diagonal canonical form. For instance,

$$\begin{aligned}
\overline{Q}_L i\gamma^\mu\partial_\mu Q_L &\to \overline{u}'_L V_u^{L\dagger} i\gamma^\mu\partial_\mu V_u^L u'_L + \overline{d}'_L V_d^{L\dagger} i\gamma^\mu\partial_\mu V_d^L d'_L \\
&= \overline{u}_L i\gamma^\mu\partial_\mu u'_L + \overline{d}'_L i\gamma^\mu\partial_\mu d'_L\ .
\end{aligned} \tag{44}$$

These keep the same form because the kinetic terms only have LL and RR pieces, and do not mix the upper and lower components of the $SU(2)_L$ doublets. As a result, we always get the combination $V_f^{L,R\dagger}V_f^{L,R} = \mathbb{I}$ in generation space. The primed field basis we have defined thus has canonical kinetic terms and diagonal masses, and is therefore a good basis for perturbation theory. It is often called the *mass eigenbasis*.

Let us turn next to the effects of diagonalizing the fermion mass matrices on the rest of the theory. By construction, or from Eq. (43), we see that the couplings of the primed fields to the Higgs boson h are all generation-diagonal. The couplings of the fermions to the photon A_μ, the massive Z_μ vector, and the gluon G^a_μ are also diagonal in generation space. This comes about for exactly the same reason that the fermion kinetic terms remain diagonal – the unitary transformations cancel each other out.

Things are more interesting for the couplings of fermions to the massive $W^\pm_\mu$ vectors. Here we have

$$\begin{aligned} -\mathscr{L} &\supset \frac{g}{\sqrt{2}}\,\overline{u}_L\gamma^\mu W^+_\mu d_R + \frac{g}{\sqrt{2}}\,\overline{\nu}_L\gamma^\mu W^+_\mu e_R + (h.c.) \\ &= \frac{g}{\sqrt{2}}\,\overline{u}'_L(V_u^{L\dagger}V_d^L)\gamma^\mu W^+_\mu d'_R + \frac{g}{\sqrt{2}}\,\overline{\nu}'_L(V_\nu^{L\dagger}V_e^L)\gamma^\mu W^+_\mu e'_R + (h.c.)\ . \end{aligned} \tag{45}$$

The unitary generation-space matrix appearing in the quark term is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix,

$$V^{(CKM)} = V_u^{L\dagger}V_d^L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}\ . \tag{46}$$

This represents a physical cross-generational mixing. For the leptons, on the other hand, we can always choose the neutrino mixing matrix $V_\nu^L = V_e^L$ without changing anything else in the SM Lagrangian.[2] Thus, we can set our field variables such that the couplings of the W to leptons remain generation-diagonal. The only genuine source of flavour mixing in the SM is therefore the CKM matrix.

Generation mixing is observed experimentally and seems to be consistent with the CKM picture. Numerically, the magnitudes of the entries in the CKM matrix are [7]

$$|V^{(CKM)}| \simeq \begin{pmatrix} 0.9748 & 0.226 & 0.0041 \\ 0.220 & 0.995 & 0.042 \\ 0.0082 & 0.040 & 1.0 \end{pmatrix}\ . \tag{47}$$

The number of decimal places here corresponds to the current experimental precision.

The Yukawa couplings we began with (in the gauge eigenbasis) in Eq. (40) can be complex. This leads to complex phases in the CKM matrix. In general, one can write a 3×3 unitary matrix in terms of three rotation angles ($O(3) \subset SU(3)$) and six phases. Five of these phases can be removed by field redefinitions that leave the real, diagonal form of the mass and kinetic terms unchanged. The remaining phase is physical, and gives rise to observable CP violation. We will discuss this later on in the course.

[2]This would not be true if it were possible to write a mass term for the neutrinos in the SM.

4 Computing with the Standard Model

To compute within the SM, it is standard to work in the mass eigenstate basis. To simplify the notation, we will drop the primes on these states that we had been using to distinguish them from the gauge eigenstates. It is also customary to assemble the 2-component SM fermions into 4-component objects so that we can use our tricks for γ^μ matrices. For this, we write

$$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \quad d = \begin{pmatrix} d_L \\ d_R \end{pmatrix}, \quad e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}, \quad \nu_L = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix} . \tag{48}$$

To isolate the L or R components, we apply the chiral projectors P_L and P_R to these 4-component fermions in the Feynman rules.

The propagators for the SM fermions and the Higgs boson are the same as we had before, and are shown in Fig. 1. For vectors, the propagators are (for momentum p)

$$A_\mu \to A_\nu : \qquad \frac{i}{p^2}(-\eta_{\mu\nu}) \tag{49}$$

$$G^a_\mu \to G^b_\nu : \qquad \frac{i}{p^2}\left[-\eta_{\mu\nu} + (1-\xi)p_\mu p_\nu/p^2\right]\delta^{ab} \tag{50}$$

$$Z_\mu \to Z_\nu : \qquad \frac{i}{p^2 - m_Z^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2\right) \tag{51}$$

$$W^\pm_\mu \to W^\pm_\nu : \qquad \frac{i}{p^2 - m_W^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_W^2\right) \tag{52}$$

The factor ξ in the gluon propagator depends on the choice of gauge and should cancel out in any physically observable quantity. The $W^\pm$ and Z^0 propagators correspond specifically to our choice of *unitary gauge*, and they describe the propagation of a massive vector.[3]

Spin polarization factors for external fermion lines in a Feynman diagram are identical to those we discussed for QED and general non-Abelian gauge theories. External vector lines pick up a polarization 4-vector $\epsilon^\mu(p,\lambda)$, as shown in Fig. 1, where p is the momentum of the vector and λ labels the polarization state. Massive and massless vectors have different numbers of polarization states. The massless photon and gluons have *two* physical transverse polarizations, while the massive $W^\pm$ and Z^0 vectors have three. Independent of whether a vector is massive or massless, we always have

$$\epsilon_\mu(p,\lambda)\, p^\mu = 0 . \tag{53}$$

These properties have important consequences for evaluating Feynman diagrams. In many cases we only care about the unpolarized cross-section, where the final polarizations are summed over and the initial polarizations are averaged. These spin sums can often be

[3] They take different forms in other gauges.

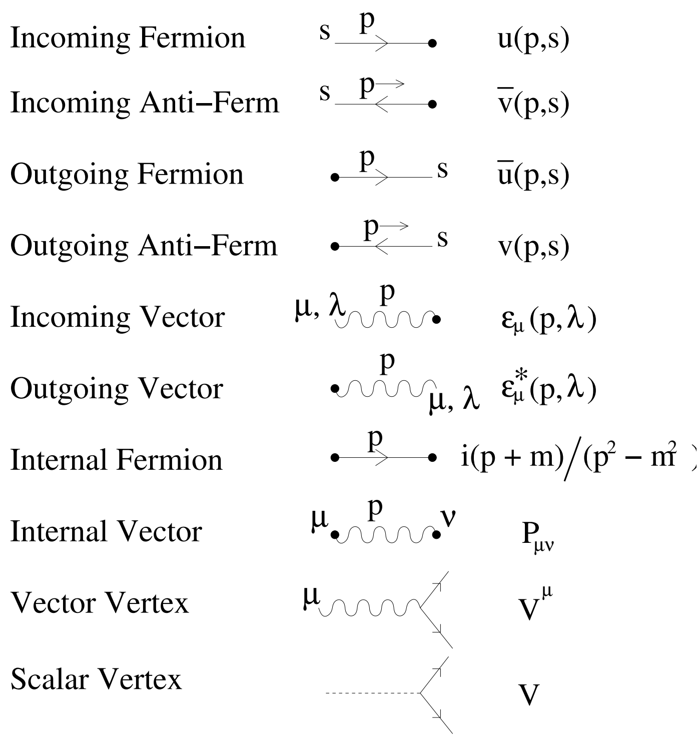

Figure 1: Feynman rules for the Standard Model.

simplified using (partial) completeness relations. For the SM vectors, we have

$$W_\mu,\ Z_\mu: \qquad \sum_{\lambda=1}^{3} \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + p_\mu p_\nu/m^2 \tag{54}$$

$$A_\mu: \qquad \sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can ignore}) \tag{55}$$

$$G_\mu: \qquad \sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can't ignore}) \tag{56}$$

The non-ignorable stuff for the gluon polarization sum is related to the presence of non-decoupling ghost fields in the theory. In this case, it usually easiest to choose an explicit set of transverse polarization vectors satisfying

$$\epsilon(p,\lambda)\cdot\epsilon^*(p,\lambda') = \delta_{\lambda,\lambda'}, \qquad p\cdot\epsilon(p,\lambda) = 0, \qquad (1,0,0,0)\cdot\epsilon(p,\lambda) = 0\ . \tag{57}$$

Of course, explicit polarizations can also be used for the other vectors.

There are lots of interaction vertices in the SM, and they are straightforward to work out from the Lagrangian. We will collect only the fermion-vector couplings here. Comparing to the general notation in Fig. 1, the fermion-photon vertex for $\psi_j \to A_\mu \psi_i$ is

$$V^\mu = -ieQ\gamma^\mu\,\delta_{ij}\ , \tag{58}$$

where i, j label the colour of the fermion ψ (with $i = j = 1$ if the fermion is uncoloured). For the gluon, the basic vertex for $\psi_j \to G^a_\mu \psi_i$ is

$$V^\mu = -ig_s \gamma^\mu t^a_{ij} \ , \tag{59}$$

where t^a_{ij} is the representation matrix corresponding to the $SU(3)_c$ rep of the fermion ψ_i. Note that for a trivial representation, $t^a_{ij} = 0$ and the vertex vanishes. The Z^0 vertex for $\psi_j \to Z_\mu \psi_i$ is

$$V^\mu = -i\bar{g}\gamma^\mu \left[(t^3 - Qs^2_W)P_L + (0 - Qs^2_W)P_R\right] \delta_{ij} \ . \tag{60}$$

Fermion projectors have been used here to isolate the different couplings of the Z^0 to the left- and right-handed components of the 4-component fermions we are working with. For the $W^\pm$, we have for $\psi_{Bj} \to W^-_\mu \psi'_{Ai}$ (where A, B are the flavour indices of the fermion)

$$V^\mu = -i\frac{g}{\sqrt{2}}\gamma^\mu P_L V^{(CKM)}_{AB} \, \delta_{ij} \ . \tag{61}$$

Note that here ψ is the lower component of an $SU(2)_L$ doublet while ψ' is an upper component. For $\psi'_A \to W^+_\mu \psi_B$ one gets the same vertex but with $V^{(CKM)\dagger}_{AB}$. Note also that except for the gluon coupling, the incoming and outgoing colour states at a vector vertex are the same, corresponding to the δ_{ij} factors. Similarly, the flavours of the incoming and outgoing fermions are identical except for the $W^\pm$ couplings and so we have not included flavour indices in the other vertices. The vertex for a fermion ψ coupling to the Higgs boson is

$$V = -i\frac{m_\psi}{\sqrt{2}v}\,\delta_{ij} \ , \tag{62}$$

where m_ψ is the fermion mass. This coupling is also diagonal in flavour (and colour) space.

e.g. 1. Amplitude for $e^+e^- \to u\bar{u}$ via the Z^0.

The Feynman diagram for this process is shown in Fig. 2. We find the amplitude

$$\begin{aligned} -i\mathcal{M} \;\; &= \;\; -i\bar{g}^2\left(\frac{1}{p^2 - m^2_Z}\right)(-\eta_{\mu\nu} + p_\mu p_\nu/m^2_Z)\,\delta_{ij} \\ &\qquad \bar{u}_3\gamma^\mu\left[(\frac{1}{2} - \frac{2}{3}s^2_W)P_L + (0 - \frac{2}{3}s^2_W)P_R\right] v_4 \\ &\qquad \bar{v}_2\gamma^\nu\left[(-\frac{1}{2} + s^2_W)P_L + (0 + s^2_W)P_R\right] u_1 \ . \end{aligned} \tag{63}$$

Here, δ_{ij} refer to the colours in the final state, $p = (p_1 + p_2) = (p_3 + p_4)$, and the spinor subscripts label their momenta (spinor indices are contracted). Squaring and summing/averaging this amplitude goes through very much like the $e^+e^- \to \mu^+\mu^-$ example we considered earlier in QED. The key difference is that we almost always want to sum over the colour states of the outgoing quarks. Since u and $\bar{u}$ must carry the same colour index (seeing as the Z^0 coupling does not modify colour), corresponding to the δ_{ij} factor,

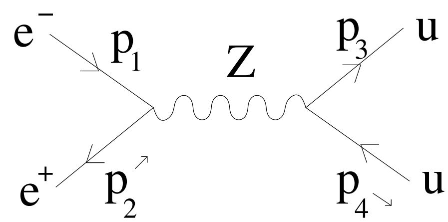

Figure 2: Diagram for $e^+e^- \to u\bar{u}$ via the Z^0.

this introduces an additional factor of $N_c = 3$ for the outgoing colour states. Note also that the full amplitude for $e^+e^- \to u\bar{u}$ gets a contribution from an intermediate photon. When $|p^2| \ll m_Z^2$, the photon contribution dominates by a factor of nearly $m_Z^2/|p^2|$. Indeed, in this case the Z^0 contribution is approximated well by simply replacing the propagator by

$$(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2)/(p^2 - m_Z^2) \to \eta_{\mu\nu}/m_Z^2 \tag{64}$$

This is the form of the vertex one would optain from a point-like interaction coupling four fermions at once.

e.g. 2. $W^+ \to u\bar{d}$.

The amplitude for this process is $(p_1 \to p_2 + p_3)$

$$-i\mathcal{M} = -i\frac{g}{\sqrt{2}}\bar{u}_2\gamma^\mu P_L V_{ud}^{(CKM)} v_3\, \epsilon_\mu(p,\lambda)\, \delta_{ij}\ . \tag{65}$$

To get the physical unpolarized rate, we should average over initial states and sum over final ones. This gives

$$\begin{aligned} \text{“}|\mathcal{M}|^{2\prime\prime} &= \frac{1}{3}\sum_{i,j}\sum_{\lambda}\sum_{s,s'}|\mathcal{M}|^2 \qquad (66)\\ &= \frac{g^2}{2}|V_{ud}|^2\left[2(p_2^\mu p_3^\alpha + p_2^\mu p_3^\alpha - p_2\cdot p_3\,\eta^{\mu\alpha}) + 2i\epsilon^{\rho\mu\sigma\alpha}p_{2_\rho}p_{3_\sigma}\right](-\eta_{\mu\alpha} + p_{1_\mu}p_{1_\alpha}/m_W^2)\\ &\simeq g^2|V_{ud}|^2 m_W^2. \end{aligned}$$

In the first line, the sums run over the colours of the quarks, the polarizations of the initial W^+ (with a 1/3 factor to make it into an average over initial pols), and a sum over final state spins. In the last line we have ignored the u and d masses which are much smaller than the W mass and correct this result by factors of $m_{u,d}^2/m_W^2$. In this approximation, the partial decay width for this channel is

$$\Gamma(W^+ \to u\bar{d}) = \frac{g^2}{8\pi}|V_{ud}|^2 m_W. \tag{67}$$

This is typical for a 2-body decay width: it goes like (mass) $\times$ (coupling)$^2/16\pi$, up to factors of order unity.

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