# P528 Notes #4: Gauge Invariance

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In the last set of notes we studied symmetries. These are transformations of a system that leave the physics unchanged. For the quantum field theories that we are interested in to describe elementary particles, this means that the action should be invariant under symmetry transformations. Among the symmetries we studied, we considered both discrete and continuous symmetries. In all cases, the corresponding transformations acted at all points in spacetime in the same way. For this reason, these symmetries are sometimes called *global symmetries*.

The topic of these notes is *gauge invariance*, which we will see corresponds to invariances under transformations can that act in a spacetime-dependent way. For this reason, they are somtimes called *local symmetries*. However, this description can be a bit misleading since the physical interpretation of gauge invariance is very different from global symmetries.

To explain all this, we will start with the familiar example of quantum electrodynamics (QED). The underlying invariance corresponds to the continuous Lie group $U(1)$, and we will see that gauge invariance dictates how the photon couples to charged matter. From there, we will generalize to gauge theories based on more complicated Lie groups, bringing us to the non-Abelian gauge theories that arise in the Standard Model.

# 1 Gauge Invariance and QED

Recall the QED Lagrangian:

$$\mathscr{L} = \sum_i \left[\,\overline{\psi}_i i\gamma^\mu(\partial_\mu + ieQ_iA_\mu)\psi_i - m\overline{\psi}_i\psi_i\,\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ . \tag{1}$$

This theory has a continuous symmetry under the transformations

$$\left\{\begin{array}{ccc} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu \end{array}\right. \tag{2}$$

This works provided the transformation parameter $\alpha$ has the same value everywhere.

Consider next what happens if we allow the transformation parameter to vary over spacetime: $\alpha = \alpha(x)$. Doing so, we find that the transformation above is no longer a symmetry of the theory. For example,

$$\overline{\psi_i}i\gamma^\mu\partial_\mu\psi_i \to \overline{\psi}_i i\gamma^\mu\partial_\mu\psi_i + \overline{\psi}i\gamma^\mu(iQ_i\partial_\mu\alpha)\psi_i\ . \tag{3}$$

Thus, the transformation of Eq. (2) is *not* a symmetry of the theory for non-constant parameters $\alpha(x)$ due to the derivative acting on it.

The invariance of the theory under spacetime-dependent transformations is restored if the vector field also transforms according to:

$$\left\{ \begin{array}{lcl} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{array} \right. \tag{4}$$

To see this, note that the combined transformations imply that

$$(\partial_\mu + ieQ_iA_\mu)\psi_i := D_\mu\psi_i \to e^{iQ_i\alpha}D_\mu\psi_i \ , \tag{5}$$

and therefore $\overline{\psi}_i i\gamma^\mu D_\mu\psi_i$ is invariant under the transformation for arbitrary $\alpha(x)$. The differential operator $D_\mu$ is sometimes called a *covariant derivative*. It is also not hard to check that this shift in the photon field does not alter the photon kinetic field:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{e}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\alpha = F_{\mu\nu} + 0 \ . \tag{6}$$

Thus, QED is invariant under the combined transformations of Eq. (4) for any reasonable arbitrary function $\alpha(x)$.

At first glance this invariance might just seem like a clever trick, but the river beneath these still waters runs deep. Thinking back to regular electromagnetism (of which QED is just the quantized version), one often deals with scalar and vector potentials. These potentials are not unique and are therefore not observable (for the most part), and the true "physical" quantities are the electric and magnetic fields. The vector field $A^\mu$ in QED, corresponding to the photon, is identified with these potentials by

$$A^\mu = (\phi, \vec{A}) \ , \tag{7}$$

where $\phi$ and $\vec{A}$ are the usual scalar and vector potentials. This is justified by the equations of motion derived from the QED Lagrangian provided we also identify

$$F^{0i} = -E^i \ , \qquad F^{ij} = -\epsilon^{ijk}B^k \ , \tag{8}$$

with the electric and magnetic fields. With this identification, the transformations of Eq. (4) coincide with the usual "gauge" transformations you should have encountered in electromagnetism. Sometimes we call $A^\mu$ the gauge boson and the operation of Eq. (4) a gauge transformation.

Keeping in mind the story from electromagnetism, the interpretation of the quantum fields in QED is that ***only those quantities that are invariant under the transformations of Eq.*** (4) ***are physically observable.*** In particular, the vector field $A_\mu$ that represents the photon is not itself an observable quantity, but the gauge-invariant field strength $F_{\mu\nu}$ is. Put another way, the field variables we are using are redundant, and the transformations of Eq. (4) represent an *equivalence relation*: any two set of fields $(\psi, A_\mu)$ related by such a transformation represent the same physical configuration. Sometimes the invariance under Eq. (4) is called a local or gauge symmetry, but it is not really a symmetry at all. A true symmetry implies that different physical configurations have the same properties. Gauge invariance is instead a statement about which configurations are physically observable.

Gauge invariance is also sensible if we consider the independent polarization states of the photon, of which there are two. The vector field $A_\mu$ represents the photon, but it clearly has four independent components. Of these, one component (corresponding to configurations of the form $A_\mu = \partial_\mu \phi$ for some scalar $\phi$) is already non-dynamical on account of the form of the vector kinetic term. Invariance under gauge transformations effectively removes the additional longitudinal polarization leaving behind only the two physical transverse polarization states. Note as well that if the photon had a mass term, $\mathscr{L} \supset m^2 A_\mu A^\mu/2$, the theory would no longer be gauge invariant. Instead, the longitudinal polarization mode would enter as physical degree of freedom. Equivalently, gauge invariance forces the photon to be massless.

In the discussion above we started with the QED Lagrangian and showed that it was gauge-invariant. However, the modern view is to take gauge invariance as the fundamental principle. Indeed, the only way we know of to write a consistent, renormalizable theory of interacting vector fields is to have an underlying gauge symmetry. For QED, we could have started with a local gauge invariance for a charged fermion field and built up the rest of the Lagrangian based on this requirement. In this context, the vector field is needed to allow us to define a sensible derivative operator on the fermion field, which involves taking a difference of two fields at different spacetime points with apparently different transformation properties, and corresponds to something called a *connection*.[1] Gauge invariance completely fixes the photon-fermion interactions, illustrating why it is so powerful. We will see shortly that gauge invariance is even more powerful when the underlying symmetry transformations are more complicated.

# 2 Non-Abelian Gauge Invariance

We have just seen that QED has an underlying invariance under $U(1)$ gauge transformations, and this invariance determines nearly the whole structure of the theory. Now, we also know that $U(1)$ is just the tip of the iceberg when it comes to compact Lie groups. From this point of view, it is completely natural to try to construct field theories with a gauge invariance under non-Abelian transformation groups such as $SU(N)$ or its many friends. This is what we will do here. Along the way, we will see that much of the structure of QED goes through unchanged, but that there are a few very important differences.

Consider an irreducible representation (= irrep) $r$ of a non-Abelian compact Lie group $G$. If the irrep has dimension $n$, we can write the representation matrices according to[2]

$$U_r \;=\; e^{i\alpha^a t^a_r} \;:=\; \sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha^a t^a_r)^n \;, \tag{9}$$

where the generators $t^a_r$ are $(n \times n)$ Hermitian matrices satisfying the Lie algebra relation

$$[t^a_r, t^b_r] = i f^{abc} t^c \;. \tag{10}$$

[1] See Ref. [2] for a nice explanation of these slightly cryptic comments.
[2] As usual, any function of a matrix should be thought of as a formal power series.

This representation acts on an $n$-dimensional vector space.

A set of $n$ fields $\psi_i$, $i = 1, 2, \ldots, n$, is said to transform under the representation $r$ if the gauge transformation law for them is

$$\begin{aligned} \psi_i \;\to\; \psi_i' &= (U_r)_{ij}\psi_j \\ &= \left(e^{i\alpha^a t^a}\right)_{ij}\psi_j \\ &= \psi_i + i\alpha^a (t_r^a)_{ij}\psi_j + \mathcal{O}(\alpha^2) \; . \end{aligned} \tag{11}$$

For the most part, we will just write the $n$ components $\psi_i$ as a single column vector $\psi$ and suppress the indices, $\psi \to U_r\psi$, but do keep in mind that they are there.

We now have that $\psi \to U_r\psi$ and $\bar{\psi} \to \bar{\psi}U_r^\dagger$. However, the derivative of $\psi$, which we will need for its kinetic term, does not transform quite so nicely if the transformation matrix varies over spacetime:

$$\partial_\mu\psi \to U_r\partial_\mu\psi + (\partial_\mu U_r)\psi \; . \tag{12}$$

It follows that $\bar{\psi}i\gamma_\mu\partial_\mu\psi$ is not invariant due to the derivative of the transformation matrix. Note that we have to be a bit careful with this piece because, in contrast to the Abelian case, one typically has

$$\partial_\mu(e^{i\alpha^a t^a}) \neq i(\partial_\mu\alpha^a t_r^a)e^{i\alpha^a t_r^a} \neq e^{i\alpha^a t_r^a}\, i(\partial_\mu\alpha^a\, t_r^a) \; . \tag{13}$$

The reason for this is that $\alpha^a t_r^a$ and $(\partial_\mu\alpha^a)t_r^a$ do not commute with each other unless $\partial_\mu\alpha^a = \lambda\alpha^a$. Thus, we will stick with the correct expression of Eq. (12).

To make the kinetic term for the charged field invariant under non-Abelian transformations of the form of Eq. (11), we mimic QED and introduce a matrix-valued vector field

$$A_{r\mu} \;:=\; A_\mu^a t_r^a \tag{14}$$

that transforms according to

$$A_{r\mu} \;\to\; U_r A_{r\mu} U_r^{-1} + \frac{1}{ig}U_r(\partial_\mu U_r^{-1}) \;:=\; \frac{1}{ig}U_r(D_\mu U_r^{-1}) \; . \tag{15}$$

Coupling this to the charged field, we get

$$\begin{aligned} (\partial_\mu + igA_{r\mu})\psi \;&\to\; \left[\mathbb{I}\,\partial_\mu + ig\left(U_r A_\mu U_r^{-1} + \frac{1}{ig}U_r\partial_\mu U_r^{-1}\right)\right]U_r\psi \\ &= U_r\left(\partial_\mu + igA_{r\mu}\right)\psi \; . \end{aligned} \tag{16}$$

As in QED, we call the combination $D_{r\mu} = (\partial_\mu + igA_{r\mu})$ the covariant derivative operator for the representation $r$,[3] and $\bar{\psi}i\gamma^\mu D_{r\mu}\psi$ (for $\psi$ a fermion) is gauge invariant.

The definitions above lead to (at least) two important questions. First, how do we construct a reasonable gauge-invariant kinetic term for the gauge field? Second, our definition

[3] The $\partial_\mu$ part of this operator is implicitly multiplied by the $n \times n$ identity matrix.

of the gauge field $A_{r\mu}$ depends on the representation of the corresponding matter field, so do we need additional gauge fields for other matter fields transforming under different representations? It turns out that the answers to both questions are closely related.

Starting with the second question, working out the transformation law explicitly to linear order in the parameter $\alpha^a$ one finds that

$$\begin{aligned} A_\mu^a t_r^a \to A'^a_{r\mu} t_r^a &= U_r A_{r\mu} U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1} \\ &= \left( A_\mu^a + f^{abc} A_\mu^b \alpha^c - \frac{1}{g} \partial_\mu \alpha^a \right) t_r^a + \mathcal{O}(\alpha^2) \ . \end{aligned} \tag{17}$$

At this order, we see that the transformation law for the coefficient fields $A_\mu^a$ is independent of the specific representation. Moreover, for $\alpha^a = constant$ the second line shows that it corresponds to $A_\mu^a$ transforming in the adjoint representation of the group. One can extend this result to all orders in $\alpha^a$ (by induction or by composing infinitesimal transformations). Therefore, it is sufficient to introduce a single set of coefficient gauge fields $A_\mu^a$ to ensure the invariance of the kinetic terms of fields transforming under any representation at all of the gauge group.

Moving next to the kinetic term for these gauge fields, a reasonable first guess would be to start with the combination $\partial_\mu A_\nu^a - \partial_\nu A_\mu^a$. Unfortunately, it has a complicated gauge transformation property and it is not at all clear how to put it into a gauge-invariant kinetic term. Instead, let us use the nice gauge transformation properties of the covariant derivative as our guide. Acting on a field transforming under the rep $r$, we found that the covariant derivative of that field transforms as $D_\mu \psi \to U_r D_\mu \psi$. Equivalently, as a differential operator, we have that $D_\mu \to U_r D_\mu U_r^{-1}$. In the same way, the covariant commutator differential operator transforms as $[D_\mu, D_\nu] \to U_r [D_\mu, D_\nu] U_r^{-1}$. Now, working out the effect of this operator on any field $\psi$ transforming in the rep $r$, one finds

$$\begin{aligned} [D_\mu, D_\nu]\psi &= ig\, t_r^a (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\, f^{abc} A_\mu^b A_\nu^c)\psi \\ &:= ig\, t_r^a F_{\mu\nu}^a \psi \ , \end{aligned} \tag{18}$$

where $F_{\mu\nu}^a$ is defined to be the expression in the first line. This result gives us precisely what we want: the commutator of these two differential operators does not involve any derivatives of $\psi$ at all, it transforms in a reasonable way, and it contains the pieces we want to build a vector boson kinetic term.

A reasonable gauge-invariant kinetic term for the gauge fields is therefore[4]

$$\begin{aligned} \mathscr{L} &\supset -\frac{1}{4(ig)^2 T_2(r)} tr([D_\mu, D_\nu][D^\mu, D^\nu]) \\ &= -\frac{1}{4(ig)^2 T_2(r)} \,(ig)^2 F_{\mu\nu}^a F^{b\,\mu\nu}\, tr(t_r^a t_r^b) \\ &= -\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu}. \end{aligned} \tag{19}$$

[4]This is gauge invariant due to the cyclicity of the trace: $tr(UMU^{-1}) = tr(M)$.

Note that even though we used a specific representation to define the gauge kinetic term, the third line of Eq. (19) shows that it can be written in a way that is representation-independent.

In all the discussion here, we have cobbled together a sensible gauge-invariant Lagrangian by fiddling around. However, a slightly more careful treatment shows that the matter-gauge couplings we have obtained are essentially unique. In other words, the requirement of gauge-invariance completely fixes the structure of the gauge interactions. As in QED, this is why it makes sense to think of gauge-invariance as the fundamental underlying feature. Also like in QED, gauge-invariance is to be treated as an equivalence relation rather than a genuine symmetry.

# 3 Computing with Non-Abelian Gauge Theories

Based on the discussion above, we are now able to write down the Lagrangian for a gauge field and a set of fermions $\psi$ transforming in a representation $r$ of the (possibly non-Abelian) gauge group $G$.[5] The kinetic and gauge-matter interaction terms are

$$\mathscr{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}i\gamma^\mu D_\mu\psi + \ldots \tag{20}$$

Additional terms can include matter-matter interactions and higher-dimensional operators provided they are consistent with gauge invariance. It is also straightforward to add other fermion species transforming under different represenations of the gauge group. The form of the covariant derivative operator implicitly depends on the representation of the field upon which it acts, and is given by

$$\psi i\gamma^\mu D_\mu\psi = \bar{\psi}i\gamma^\mu(\partial_\mu + igA^a_\mu t^a_r)\psi, \tag{21}$$

where $t^a_r$ are the generators of the representation $r$ under which the field $\psi$ transforms. Note that $t^a_r = 0$ when $r$ is the trivial representation. This implies that a field transforming under the trivial representation of a gauge group does not couple to the gauge boson, and is said to be uncharged. Expanding out the gauge kinetic term, one obtains

$$\begin{aligned} -\frac{1}{4}(F^a_{\mu\nu})^2 &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \\ &\quad + \frac{1}{2}g\, f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}. \end{aligned} \tag{22}$$

The first term is evidently the kinetic term for the vector, while the second two terms are gauge boson self-interactions induced by the non-Abelian nature of the gauge group; this is perhaps the most important consequence of having a non-Abelian group. These expressions also reduce to the Abelian case if we set $f^{abc} \to 0$ and $t^a_r \to Q$, where $Q$ is the $U(1)$ charge of the field $\psi$.

Starting from Eq. (21) and Eq. (22) we can derive all the Feynman rules for gauge interactions in a non-Abelian gauge theory. The final result is very nearly identical to

[5] In the homework you will learn how to add charged scalars to the theory.

QED with some additional group theoretic factors for decoration. However, there are a few important differences that must be taken into account.

In order to obtain a sensible quantum propagator for the gauge field, it is usually necessary to choose a specific gauge.[6] A very popular family of gauge choices goes by the name of $R_\xi$, with each choice in the family characterized by a constant parameter $\xi$. This leads to a vector propagator for $A^a_\mu \to A^b_\nu$ of

$$D^{ab}_{\mu\nu}(p) = \frac{i}{p^2}\,\delta^{ab}\left[-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{23}$$

The corresponding Feynman rule is shown in Fig. 1. Some popular $\xi$ values are the Landau gauge with $\xi = 0$, and the Feynman-'t Hooft gauge with $\xi = 1$. Any observable quantity must be independent of $\xi$ due to the requirement of gauge invariance. This can be a useful way to check complicated calculations.

A full quantum derivation of the $R_\xi$-gauge propagator of Eq. (23) also leads one to include an additional set of massless *Faddeev-Popov ghost* fields transforming under the adjoint rep of the gauge group. Ghost fields have the unusual property of being anti-commuting Lorentz scalars. (Typically, even-spin fields (bosons) are commuting while odd-spin fields (fermions) are anti-commuting.) They do not represent physical particle excitations.[7] Instead, ghost fields play the role of "negative degrees of freedom" in Feynman diagram calculations to cancel off the gauge redundancy of the vector gauge fields. In practice, this means that ghost fields only appear as intermediate states in loop diagrams, and never appear as on-shell external states in a physical process. With one additional minor requirement to be discussed below, this implies that we can completely ignore the ghost fields as long as we stick to tree-level processes.

The propagator of a fermion field is nearly identical to the QED case. For $\psi_i \to \psi_j$ (where $i$ and $j$ are the indices of the rep), we have

$$D_{ij}(p) = \delta_{ij}\frac{i(\not{p}+m)}{p^2-m^2}. \tag{24}$$

Similarly, for a charged complex scalar $\phi_i \to \phi_j$,

$$D_{ij}(p) = \delta_{ij}\frac{i}{p^2-m^2}. \tag{25}$$

We illustrate the corresponding diagrams in Fig. 1.

Vertex factors are straightforward to derive from Eqs. (21,22). The vertex corresponding to the fermion-vector $\psi_j \to \psi_i + A^a_\mu$ interaction is

$$V_{ffG} = -ig\,(t^a_r)_{ij}\gamma_\mu. \tag{26}$$

[6] The same is true for QED, but there we can get away with ignoring the implications when computing Feynman diagrams for some slightly subtle reasons.

[7] Ghosts also come up in QED and other Abelian gauge theories, but since the adjoint rep of such theories is trivial, they do not couple to anything and can be ignored when computing Feynman diagrams.

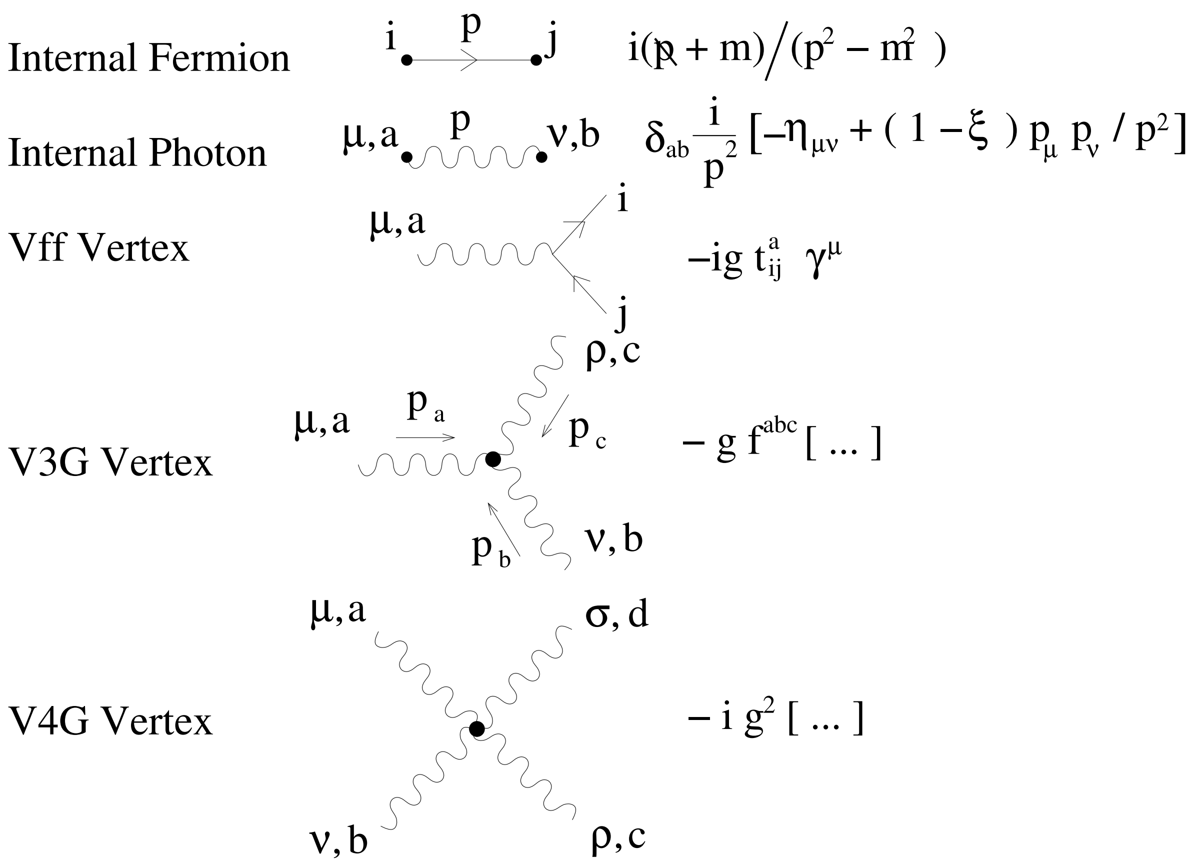


Figure 1: Feynman rules for a non-Abelian gauge theory.

Notice how the indices on the generator matrix match up with the representation indices on the fermions. There are also three- and four-point gauge self interactions. For $A^a_\mu A^b_\nu A^c_\rho$ we have

$$V_{3G} = -g\, f^{abc}\left[\eta^{\mu\nu}(p_a - p_b)^\rho + \eta^{\nu\rho}(p_b - p_c)^\mu + \eta^{\rho\mu}(p_c - p_a)^\nu\right], \tag{27}$$

where $p_{a,b,c}$ are the *incoming* momenta carried by the vectors $A^{a,b,c}_{\mu,\nu,\rho}$ at the vertex (see Fig. 1). For the $A^a_\mu A^b_\nu A^c_\rho A^d_\sigma$ vertex we get

$$\begin{aligned} V_{4G} \quad = \quad & -ig^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})]. \end{aligned} \tag{28}$$

Again, take a look at Fig. 1.

Feynman diagram calculations in non-Abelian gauge theories are very similar to those in QED up to some additional group theoretic factors and the gauge field self-interactions. As in QED, one builds up an amplitude by writing down all the Feynman diagrams for the process. Each diagram has a numerical value which is constructed by tracing backwards along fermion lines, putting in internal propagators and vertex factors, and adding the initial- and final-state polarization vectors and fermion spinors. The main complication is that one must also keep track of non-Abelian group theory factors. For example, each gauge boson line has an adjoint index associated to it while each charged fermion or scalar line transforming in the rep $r$ has an index corresponding that rep.

The amplitude that is computed has specific external spin states, vector polarizations, and group theory values. In most cases we want unpolarized cross-sections that are summed over all distinct final states and averaged over all distinct initial states. The fermion spin and vector polarization parts are nearly identical to QED, but now we also have to sum over the distinct states in a given rep. For example, the amplitude for a process involving $\psi_j + X \to \psi_i + Y$ will take the form $\mathcal{M}_{ij}$, where $i$ and $j$ are indices for the rep of $\psi$. The squared and summed matrix element for the process will then involve

$$\text{“}|\mathcal{M}|^{2\text{''}} = \frac{1}{d(r)} \sum_{i,j} \mathcal{M}_{ij}^* \mathcal{M}_{ij}, \tag{29}$$

where $d(r)$ is the dimension of the rep of $\psi$.

***e.g. 1*** $\psi\bar{\psi} \to \chi\bar{\chi}$

Suppose $\psi$ and $\chi$ are massless fermions transforming under the reps $r_\psi$ and $r_\chi$ of the non-Abelian gauge group $G$. The leading Feynman diagram for this process is given in Fig. 2. The amplitude is

$$-i\mathcal{M} = -ig^2 (t^a_{r_\psi})_{ij} (t^b_{r_\chi})_{pq} \frac{\delta_{ab}}{p^2} (\bar{u}_3 \gamma^\mu v_4)(\bar{v}_2 \gamma^\nu u_1) \left[-\eta_{\mu\nu} + (1-\xi) p_\mu p_\nu / p^2\right]. \tag{30}$$

Here, $p = (p_1 + p_2) = (p_3 + p_4)$, and the subscripts label the momenta of the spinors (with spinor indices contracted). Squaring and summing/averaging the matrix element, the spin part comes out just like in the $(e^+e^- \to \mu^+\mu^-)$ example we looked into previously in QED (and one finds that the $\xi$-dependent part does not contribute in the end). There is, however, a new group theory piece. The net result is

$$\text{“}|\mathcal{M}|^{2\text{''}} = (GT)\, \frac{8g^4}{p^4} [(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] \tag{31}$$

with the group theory factor given by

$$\begin{aligned} (GT) &= \frac{1}{d^2(r_\psi)} \sum_{i,j} \sum_{p,q} (t^a_{r_\psi})^*_{ij} (t^c_{r_\psi})_{ij} (t^b_{r_\chi})^*_{pq} (t^d_{r_\chi})_{pq}\, \delta_{ab}\delta_{cd} \\ &= \frac{1}{d^2(r_\psi)}\, tr(t^a_{r_\psi} t^c_{r_\psi})\, tr(t^a_{r_\chi} t^c_{r_\chi})\, \delta_{ab}\delta_{cd} \\ &= \frac{d(A)}{d^2(r_\psi)} T_2(r_\psi) T_2(r_\chi). \end{aligned} \tag{32}$$

In the second line we have made use of the Hermiticity of the $t^a$ to write $(t^a)^* = (t^a)^t$ while in the third line we have used $\delta^{ac}\delta^{bd}\delta_{ab}\delta_{cd} = \delta^c{}_c = d(A)$. A useful trick for obtaining the gauge matrix factors is to trace backwards along the “gauge flow” in the diagram, much like one traces backwards along fermion lines to get the spinor factors.

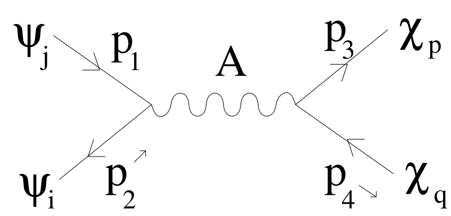


Figure 2: The leading Feynman diagram for $\psi\bar{\psi} \to \chi\bar{\chi}$.

Relative to QED (or other purely Abelian gauge theories), there is one additional complication related to the polarization of external vector states. Recall that in QED, we were able to use a polarization completeness relation to simplify the polarization sums:

$$\sum_\lambda \epsilon^*_\mu(p,\lambda)\epsilon_\nu(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can ignore}) \qquad (\text{Abelian case})\ . \tag{33}$$

The "extra stuff" here is related to the fact that there are only two distinct physical polarizations for a massless vector, whereas four states would be needed for full completeness, corresponding to a sum that produces $\eta_{\mu\nu}$ alone. Fortunately, in Abelian gauge theories the extra stuff always vanishes automatically when it is contracted with a physical amplitude and can therefore be neglected. In the non-Abelian case it turns out that you can't always get away with ignoring the extra stuff. There are various ways of handling this, but in many cases the easiest is to specify explicitly the two transverse polarization vectors $\epsilon^\mu(\vec{p},\lambda)$, $\lambda = 1,2$, and sum over them. You can choose these however you want provided they satisfy the conditions

$$\epsilon^*(\vec{p},\lambda)\cdot\epsilon(\vec{p},\lambda') = -\delta_{\lambda\lambda'}, \qquad (1,\vec{0})\cdot\epsilon(p,\lambda) = 0, \qquad p\cdot\epsilon(\vec{p},\lambda) = 0. \tag{34}$$

For example, if $\vec{p} = p\hat{z}$, two popular choices are $\{(0,1,0,0),\ (0,0,1,0)\}$ (linear polarizations) and $\{(0,1,i,0)/\sqrt{2},\ (0,1,-i,0)/\sqrt{2}\}$ (right- and left-handed circular polarizations). Note as well that gauge invariance implies that the vector boson is massless, so $p$ is a lightlike 4-vector with $p^2 = 0$ (as a 4-vector).

# 4 The Fundamental QCD Lagrangian

Quantum chromodynamics is the underlying theory of the strong force. It is a non-Abelian gauge theory with gauge group $SU(3)$. The gauge fields (of which there are $8 = 3^2 - 1$ components) are called gluons $G^a_\mu$. In addition, there are six fermionic quark fields, $q = u,\ d,\ c,\ s,\ t,\ b$, each transforming under the fundamental $\mathbf{3}$ representation of $SU(3)$. The different quark fields are called *flavours*. For each flavour, the three components of the $\mathbf{3}$ representation are called colours: $q = q_i$, $i = 1,2,3$. From the point of view of QCD, there is nothing terribly fundamental about flavour while the colours are an essential part of the underlying gauge symmetry structure. The terminology of flavour and colour is also frequently applied to other non-Abelian gauge theories.

Given what we know about non-Abelian gauge theories, we can write down the QCD Lagrangian immediately:

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \sum_{q=u,d,c,s,t,b} \bar{q}(i\gamma^\mu D_\mu - m_q)q, \tag{35}$$

where $D_\mu = (\partial_\mu + ig_s t^a G^a_\mu)$ and $m_q$ is the mass of quark $q$. That's it!

# References


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