PHYS 528 Homework #7

Due: March 15, 2024, 12pm PDT

1. Dirac and Majorana fermions.

   a) The electron in QED is a Dirac fermion consisting of independent e_L and e_R LH and RH 2-component pieces (and their conjugates). How do these transform under $U(1)_{em}$ gauge transformations? Rewrite both as 2-component LH fermion fields (and their conjugates). How do these two independent LH fermion fields transform under $U(1)_{em}$?

   b) A 4-component fermion ψ is said to be *Majorana* if its two 2-component pieces are derived from the **same** 2-component LH fermion. Show that this implies that (up to a possible phase)

   $$\psi^c = \psi\ , \qquad \text{where} \qquad \psi^c = i\gamma^2\gamma^0(\bar{\psi})^t\ .$$

   c) A *Majorana mass* for a general 4-component fermion ψ is defined to be

   $$\mathscr{L} \supset -\frac{1}{2}M\,\overline{(\psi^c)}\psi\ .$$

   Express this operator in terms of the 2-component pieces of $\psi = (\chi, \bar{\xi})^t$.
   Note: do not assume that ψ is a Majorana fermion.

   d) Prove that a Majorana mass is forbidden if the theory has a $U(1)$ symmetry under which ψ transforms non-trivially.

   e) A pair of LH 2-component fermions χ and ξ are said to be *vector-like* if they transform under conjugate unitary representations of the underlying (non-Lorentz) symmetry group of the theory relative to each other. (*i.e.* $\chi \to e^{i\beta^a t^a_r}\chi$, $\xi \to e^{-i\beta^a t^{a*}_r}\xi$ with Hermitian t^a_r, real β^a.) Show that this allows a standard Dirac mass term $m\bar{\psi}\psi$ for the 4-component fermion $\psi = (\chi, \bar{\xi})$.

2. Z decays in the SM.

   Compute the partial decay widths of the Z^0 into pairs of each of the SM fermions. You may ignore the fermion mass whenever it is smaller than one tenth the Z^0 mass: $m_f < m_Z/10$. Note that the branching fraction of a particular decay mode with partial width Γ_i is $\mathrm{BR}_i = \Gamma_i/\Gamma_{tot}$. Plug in numbers and compare to data.
   Hint: data can be found at http://pdglive.lbl.gov/.
   Hint: there are lots of terms, but many of them vanish!
   Hint: $P_LP_R = 0$, $P_L^2 = P_L$, $P_L\gamma^\mu = \gamma^\mu P_R$, $tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$.
   Hint: an $\epsilon^{\mu\nu\rho\sigma}$ term can only be non-zero if it connects with ***four*** *independent 4-vectors.*
   Hint: don't forget to take colours into account.
   Note: decays to the light u, d, and s quarks can't be distinguished from each other experimentally. For these, compare to the data for $(u\bar{u} + d\bar{d} + s\bar{s})/3$.
   Note: neutrinos count as "invisible" here.

3. Invisible Z decays.

   Suppose the SM contains a new "invisible" LH chiral fermion with a coupling strength g_L to the Z^0. (That is, the Lagrangian interaction is $\bar{\psi}_L \gamma^\mu Z_\mu g_L \psi_L$.) Derive the upper limit on the size of the coupling g_L as a function of the fermion mass m_ψ from the requirement that its contribution to the invisible decay width of the Z^0 be less than $\Delta\Gamma_{inv} < 2$ MeV.
   Hint: you should be able to reuse the calculation from question #2.