PHYS 528 Homework #5

Due: Mar.4, 2021

1. Scalar expansions.

   Consider a general theory of n real scalars ϕ_i $(i = 1, \ldots, n)$ with Lagrangian

   $$\mathscr{L} = \frac{1}{2}\sum_{i=1}^{n}(\partial\phi_i)^2 - V(\phi_i) \ .$$

   If $\{\phi_i = \langle\phi_i\rangle\}$ is a minimum of the potential, show that the fields $h_i(x) = \phi_i(x) - \langle\phi_i\rangle$ have canonical normalization and vanish at the minimum of the potential. Next, expand the potential about the minimum in a power series in h_i and show that $\partial^2 V/\partial\phi_i\partial\phi_j|_{\langle\phi\rangle}$ is the mass matrix for the scalars h_i.

2. SSB and NGBs.

   a) In the spontaneously broken global $U(1)$ theory discussed in the notes, work out the kinetic term and potential in terms of the new polar field variables we defined.

   b) Suppose we have the same theory but with $-\mu^2 \to +\mu^2$ in the potential. Is there still spontaneous symmetry breaking? What are the particle masses of all the real scalar degrees of freedom? *Hint: we really want canonical kinetic terms!*

   c) For the same theory, expanded around the vacuum $\langle\phi\rangle = e^{i\beta}\,v$, work out the mass matrix $\partial^2 V/\partial\phi_i\partial\phi_j|$ in terms of the original field variables ϕ and ϕ^* and show that it has a zero determinant. Also, find $F_i^a(\langle\phi\rangle)$ for an infinitesimal phase rotation and show that it is a zero eigenvalue of this mass matrix.
   Hint: treat ϕ and ϕ^* as independent degrees of freedom.

   d) For the global $SU(2)\times U(1)$-symmetric theory discussed in the notes, work out the full Lagrangian in terms of the new field variables

   $$\begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \begin{pmatrix} (\phi_{+r} + i\phi_{+i})/\sqrt{2} \\ v + (\phi_{0r} + i\phi_{0i})/\sqrt{2} \end{pmatrix}$$

   This choice corresponds to an expansion about the vacuum with $\alpha^a = \beta = 0$. What are the mass eigenvalues?

3. A semi-realistic Higgs.

   a) Starting from the global $SU(2)\times U(1)$-symmetric theory discussed in the notes, elevate this to a theory that is invariant under local $SU(2)\times U(1)$ transformations by adding an appropriate set of vector gauge fields and couplings. What is the corresponding Lagrangian?
   Hint: each gauge factor has its own gauge field and its own gauge coupling.

b) Work out the commutation relations of the modified set of generators $\tilde{t}$ and $\{t^B_{G/H}\}$ discussed in the notes.

c) Expand this theory around the vacuum after making a nice choice of gauge. Find the masses of all the physical scalars, and make sure their kinetic terms are canonical.
Hint: for the choice of gauge, start with the field expansion used in the global $SU(2) \times U(1)$ theory discussed in the notes and then simplify it enormously by choosing a gauge in analogy to what was done in the gauged $U(1)$ theory discussed there. Talk to me if you aren't sure how to proceed.

d) (**Optional - will not be marked!**) Expand the gauge-covariant kinetic term of the scalar field to find mass terms for the vector fields.
Hint: remember that $t^a = \sigma^a/2$ and compute the covariant derivative $D_\mu\phi$ as a two-component column vector. Then use the fact that the gauge fields are real to simplify its square. Also, recall that any 2×2 symmetric matrix can be diagonalized by an orthogonal matrix, and this matrix can be built from the orthonormalized eigenvectors.