PHYS 528 Homework #4

Due: Feb.16, 2024, 12pm PST

1. Scalar decay to vectors.

   Consider the interaction

$$-\mathscr{L} \supset \frac{1}{\Lambda}\, h\, V_{\mu\nu} V^{\mu\nu} \ , \tag{1}$$

   where h is a real scalar of mass m_h, $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ for some vector boson V_μ, and $\Lambda \gg m_h$ is some very large mass scale. This interaction allows the decay $h(p) \to V_\mu(k_1) + V_\nu(k_2)$, for which the amplitude is

$$-i\mathcal{M} = -\frac{2i}{\Lambda}(k_1^\mu \epsilon_1^\nu - k_1^\nu \epsilon_1^\mu)(k_2^\alpha \epsilon_2^\beta - k_2^\beta \epsilon_2^\alpha)\eta_{\alpha\mu}\eta_{\beta\nu} \ , \tag{2}$$

   where ϵ_1 and ϵ_2 refer to the polarizations of two outgoing vectors.

   a) If V_μ is massless, there are two physical polarization states and a built-in gauge invariance. Compute the summed and squared matrix element "$|\mathcal{M}|^{2}$" relevant for the total unpolarized decay rate in the h rest frame using the partial completeness relation $\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{\nu\,*}(k,\lambda) = -\eta^{\mu\nu} + (\mathit{stuff\ you\ can\ ignore})$, just like what we used for external photons in QED.

   b) A second way to compute the summed and squared matrix element "$|\mathcal{M}|^{2}$" is to specify external polarization vectors and add up the results. Do this here using the explicit polarization vectors discussed in `notes-04` and summing over all the possibilities.
   Hint: since the initial state is at rest and has no spin, you can choose the $\hat{z}$ axis to lie along the direction of the first outgoing vector, $\vec{k}_1 = \|\vec{k}_1\|\,\hat{z}$.

   c) Suppose instead that the vector V_μ is massive, with mass m_V. This implies that it has three physical polarization states. The corresponding polarization 4-vectors should satisfy

$$\epsilon(k,\lambda)\cdot\epsilon^*(k,\lambda') = -\delta_{\lambda\lambda'} \ , \qquad\qquad k\cdot\epsilon(k,\lambda) = 0 \ . \tag{3}$$

   For $\vec{k} = \|\vec{k}\|\,\hat{z}$, find a set of three 4-vectors that satisfy these conditions. You should be able to identify two of them as *transverse*, and one as *longitudinal*.

   d) Use these three polarization 4-vectors to compute the summed and squared matrix element "$|\mathcal{M}|^{2}$" for $h \to V_\mu V_\nu$ in the rest frame of the decaying scalar. Also, compare the squared matrix element for longitudinal final states to those for transverse final states.

2. Global symmetries and QED.

   Consider a universe in which there is a photon, an electron, a muon, as well as N additional Dirac fermions ψ_i with non-zero masses m_i and charges Q_i, $i = 1, 2, \ldots N$.

   a) Write Lagrangian terms for the new fermions that include canonical kinetic terms, mass terms, and consistent (gauge invariant) couplings to the photon.

   b) Suppose the theory also has a global $SU(N)$ symmetry under which the electron and muon are singlets and the N new fermions transform amongst themselves under the fundamental irrep of $SU(N)$. What does the symmetry imply for the masses and QED charges of the new fermions? Write out the resulting Lagrangian, updating your result from part a).
   Hint: it might be illuminating to write the masses and charges of the fermions in the form of $N \times N$ matrices.

   c) Compute the total *inclusive* cross section for the production of the new fermions (assuming an $SU(N)$ symmetry) from energetic electron-positron collisions in the CM frame, where *inclusive* means that you should sum over all the new fermion types in the final state. Also, keep the masses of the new fermions in your calculation but neglect the electron mass and write the answer in terms of the fermion masses and charges and the Mandelstam variable $s = (p_1 + p_2)^2$.
   Hint: you should be able to almost completely reuse the answer to question #1 on `hw-02`, so very little additional computation is needed!

   d) Starting with your result from part b), upgrade the Lagrangian in the minimal possible way such that the theory has an $SU(N)$ gauge invariance with a corresponding set of new vector bosons G^a_μ.
   Hint: vector boson kinetic terms and covariant derivatives!

3. Non-Abelian gauge invariance.

   a) Work out the details and show explicitly that the covariant derivative we discussed for the non-Abelian case transforms according to $D_\mu\psi \to U_r(D_\mu\psi)$.
   Hint: $0 = \partial_\mu(\mathbb{I}) = \partial_\mu(U_r U_r^{-1}) = \partial_\mu(U_r^{-1} U_r)$ *with* $U_r^{-1} = U_r^\dagger$.

   b) We had that $A_{r\,\mu} := A^a_\mu t^a_r \to A'^{\,a}_{r\,\mu} t^a_r = U_r A_{r\,\mu} U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1}$. For $U_r = e^{i\alpha^a t^a_r}$, work out the corresponding transformation of the coefficient functions A^a_μ to linear order in the α^a parameters to derive the result of Eq. (17) in `notes-04` explicitly. Does this result depend on the specific representation chosen? (*i.e.* would the same transformation of the A^a_μ coefficients also work for other representations?)

   c) Fill in the details of the derivation of $[D_\mu, D_\nu]\psi = igt^a_r(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu)\psi$, for ψ transforming under the rep r of the gauge group.

   d) Write out the covariant derivative acting on a field transforming under the adjoint rep of the non-Abelian group G in terms of the structure constants f^{abc}.