P528 Notes #3: Symmetries

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March 10, 2024

Symmetries play a central role in modern particle physics. Insofar as we believe that elementary particles can be described by QFT (and the evidence so far points in this direction), our role as theoretical and experimental particle physicists is to figure out the Lagrangian of our world. In particular, we must specify a set of fields and their interactions. Once we have a candidate Lagrangian, we can compute the dynamics of the theory and compare to experiment. Symmetries make the task of figuring out the Lagrangian much easier because they strongly constrain the set of possible fields and interactions. They are also enormously useful in computing the dynamics because they relate different sets of solutions to the system.

In this note, we will first describe how symmetries are dealt with in QFT. Next, we will generalize the notion of symmetries to gauge invariance as it applies to QED. And finally, we will give some formal details of the mathematical description of symmetries that will be useful later on.

1 Symmetries in QFT

Let's begin with the treatment of symmetries in QFT. We will show that if a system has a continuous symmetry, there is a related conservation law. Before getting to this key result, called *Noether's theorem*, we will start with a few examples.

e.g. 1. Discrete scalar field symmetry

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ .$$

This theory is symmetric under $\phi \to -\phi$ in that the form of the Lagrangian stays the same after the transformation. The implication of this symmetry is that for any process, the number of particles in the initial state minus the number in the final state must be even. Note that $\phi \to -\phi$ would not be a symmetry of the theory if we were to add a cubic $A\phi^3$ term to the Lagrangian.

e.g. 2. Discrete chiral symmetry

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - y\phi\bar{\psi}\psi \ .$$

This theory is symmetric under the simultaneous transformations $\phi \to -\phi$ and $\psi \to \gamma^5\psi$. (Note that the second condition implies $\bar{\psi} \to -\bar{\psi}\gamma^5$.) This symmetry is only possible if there is no fermion mass term. Such symmetries are sometimes called *chiral* symmetries.

e.g. 3. Continuous field symmetry

$$\mathscr{L} = |\partial\phi|^2 - M^2|\phi|^2 + \sum_{i=1}^{2} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - (y\,\phi\bar{\psi}_1\psi_2 + h.c.)\ .$$

This theory is symmetric under $\psi_1 \to e^{i\alpha Q_1}\psi_1$, $\psi_2 \to e^{i\alpha Q_2}\psi_2$, $\phi \to e^{i\alpha Q_\phi}\phi$ for any real constant α provided $(Q_\phi - Q_1 + Q_2) = 0$. These Q's are sometimes called the *charges* of the fields under the symmetry. In contrast to the previous examples, this symmetry is *continuous* rather than *discrete* in that it holds for any value of real parameter α.

The definition of a symmetry is a transformation of the system that the leaves the physics the same. For a field theory defined by an action that depends on a set of fields, this will be the case if and only if the transformed action has the same functional form as the original action. More precisely, for an action $S[\phi]$ that depends on the set of fields $\{\phi_i\}$ and the transformation

$$\phi_i \to \phi_i' = f_i(\phi)\ , \qquad S[\phi] \to S[\phi'] := S'[\phi]\ , \tag{1}$$

we must have $S'[\phi] = S[\phi]$ for this to be a symmetry. Equivalently, the transformed Lagrangian must have the same form as the original Lagrangian up to total derivatives. Some additional discussion of this is given in Refs. [1, 2].

Continuous symmetries are especially interesting because they imply conservation laws. This relationship is called *Noether's theorem*. Before deriving it, it is worth remembering how one obtains the classical equations of motion for a system from the action that defines it. The solution to the equation of motion is the field configuration such that for any arbitrary set of infinitesimal field variations that vanish on the boundary, $\phi_i \to \phi + \delta\phi_i$, the numerical value of the action remains the same to leading order: $\delta S = 0$. For a local field theory action, this leads to

$$\delta S = \int d^4x\, \sum_i \left(\left[\frac{\partial\mathscr{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i + \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right]\right) . \tag{2}$$

The second term above is zero automatically because the field variation is assumed to vanish on the boundary. Thus, demanding that $\delta S = 0$ for any small variation $\delta\phi_i$ implies that the bracketed quantity in the first term vanishes. These are the equations of motion for ϕ_i.

Consider now a continuous symmetry transformation parametrized by the dimensionless real variable α. For very small $|\alpha| \ll 1$, the leading variation in field ϕ_i can be written as

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \alpha\Delta_i(\phi)\ . \tag{3}$$

Since this is a symmetry, we must have to leading order in α

$$\mathscr{L}(\phi') = \mathscr{L}'(\phi) = \mathscr{L}(\phi) + \alpha\,\partial_\mu K^\mu\ , \tag{4}$$

for some K^μ. This is just an explicit statement that the transformed Lagrangian keeps the same form up to a possible total derivative (*i.e.* $\alpha\,\partial_\mu K^\mu$). Plugging the form of Eq. (3) into Eq. (4) and expanding to linear order, we find

$$\begin{aligned}
\alpha\partial_\mu K^\mu &= \mathscr{L}'(\phi) - \mathscr{L}(\phi) \qquad (5)\\
&= \sum_i \left[\frac{\partial\mathscr{L}}{\partial\phi_i}\alpha\Delta_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\alpha\Delta_i)\right]\\
&= \alpha\sum_i\left(\left[\frac{\partial\mathscr{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right]\Delta_i + \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\Delta_i\right]\right) .
\end{aligned}$$

The first term above vanishes by the equation of motion, while the second term remains. Rearranging and removing the common factor of α, this implies

$$0 = \partial_\mu j^\mu \ , \qquad \text{where} \ \ j^\mu := \left[\sum_i \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\Delta_i - K^\mu\right] . \qquad (6)$$

This result is *Noether's theorem.*

The 4-vector j^μ obtained from Noether's theorem is said to be a *conserved current.* Noether's theorem implies that there exists such a conserved current for every continuous symmetry of the system. The current is said to be conserved because if we define the conserved charge (not necessarily electric charge!) by

$$Q = \int d^3x\, j^0 \ , \qquad (7)$$

we find that

$$\partial_t Q = \int d^3x\, \partial_t j^0 = \int d^3x\, \vec{\nabla}\cdot\vec{j} = 0 \ . \qquad (8)$$

Note that we get zero because everything vanishes on the boundary, by assumption. The physical interpretation of $j^\mu = (j^0, \vec{j})$ is that j^0 is a charge density and $\vec{j}$ is a current density.

It should be mentioned that everything we have done applies to the classical field theory defined by the action. However, these results also apply to the QFT derived from the action when they are interpreted as *operator equations*, meaning that the equations of motion and conservation laws are satisfied as operator expectation values between any states (up to a few subtleties). In one-particle quantum mechanics, the corresponding relation is known as Ehrenfest's theorem.

e.g. 4. A two-field example

Consider a theory with two real fields ϕ_1 and ϕ_2:

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right)\\
&= \frac{1}{2}(\partial\phi)^t(\partial\phi) - \frac{1}{2}m^2\phi^t\phi \ ,
\end{aligned}$$

where $\phi = (\phi_1, \phi_2)^t$. This theory is symmetric under transformations of the form

$$\phi \to \phi' = \mathcal{O}\phi \ , \tag{9}$$

where $\mathcal{O}$ is any 2×2 orthogonal matrix – satisfying $\mathcal{O}^t\mathcal{O} = \mathbb{I}$. Up to a few signs, any such matrix can be parametrized in terms of the single parameter α:

$$\mathcal{O} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \ . \tag{10}$$

Applying this transformation to the Lagrangian, we find that $\mathscr{L}(\phi') = \mathscr{L}(\phi)$, and thus $K^\mu = 0$. For small rotation angles α, the transformation becomes

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 + \alpha\Delta_1 \\ \phi_2 + \alpha\Delta_2 \end{pmatrix} \ . \tag{11}$$

Thus, $\Delta_1 = -\phi_2$ and $\Delta_2 = \phi_1$. The conserved current is therefore

$$j^\mu = -(\partial^\mu\phi_1)\phi_2 + \phi_1(\partial^\mu\phi_2) \ . \tag{12}$$

It is straightforward to check that this current is indeed conserved, $\partial_\mu j^\mu = 0$.

e.g. 5. Complex scalar field

Recall that the theory of *e.g. 4.* can be rewritten in terms of a single complex scalar $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The Lagrangian becomes

$$\mathscr{L} = |\partial\phi|^2 - m^2|\phi|^2 \ . \tag{13}$$

The symmetry transformation above can now be written in the form

$$\phi \to \phi' = e^{i\alpha}\phi \ , \qquad \phi^* \to \phi'^* = e^{-i\alpha}\phi^* \ . \tag{14}$$

(Note that ϕ and ϕ^* should be thought of as independent "real" variables.) Specializing to infinitesimal α, we have

$$\Delta = i\phi \ , \qquad \Delta^* = -i\phi^* \ . \tag{15}$$

This gives the current

$$j_\mu = -i\,\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi \tag{16}$$

where $\overset{\leftrightarrow}{\partial}_\mu = (\overset{\rightarrow}{\partial}_\mu - \overset{\leftarrow}{\partial}_\mu)$

e.g. 6. Dirac fermion

Consider

$$\mathscr{L} = \overline{\psi} i\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \ . \tag{17}$$

This theory has a symmetry under

$$\psi \to e^{iQ\alpha}\psi \ , \qquad \overline{\psi} \to e^{-iQ\alpha}\overline{\psi} \ , \tag{18}$$

corresponding to $\Delta = iQ\psi$. The corresponding conserved current is

$$j^\mu = -Q\,\overline{\psi}\gamma^\mu\psi \ . \tag{19}$$

e.g. 6. Shifts

Any theory whose Lagrangian depends only on derivatives of a field has a symmetry under shifts of that field. For example,

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 \ , \qquad \phi \to \phi + \alpha \ . \tag{20}$$

In this case, $\Delta = 1$, and the current is

$$j^\mu = \eta^{\mu\nu}\partial_\nu\phi \ . \tag{21}$$

This kind of symmetry will be important for theories in which a process called *spontaneous symmetry breaking* occurs. In contrast to the previous symmetries, it is *non-linear* in the sense that the transformed field is not quite a linear combination of the original field.

2 Symmetries and Groups

To go beyond QED, it will be useful to have a more formal mathematical description of symmetry transformations. In particular, symmetry transformations obey the mathematical properties of a *group*, and it is worth spending a bit of time discussing them.[1] Along the way, we will be led to introduce the concepts of *representations* and *Lie groups*.

A group G is a set of objects together with a multiplication rule such that:

1. if $f,\, g \in G$ then $h = f\cdot g \in G$ (closure)
2. $f\cdot(g\cdot h) = (f\cdot g)\cdot h$ (associativity)
3. there exists an identity element $1 \in G$ with $1\cdot f = f\cdot 1 = f$ for any $f \in G$ (identity)
4. for every $f \in G$ there exists an inverse element f^{-1} such that $f\cdot f^{-1} = f^{-1}\cdot f = 1$ (invertibility)

A group can be defined via a multiplication table that specifies the value of $f\cdot g$ for every pair of elements $f,\, g \in G$. An *Abelian* group is one for which $f\cdot g = g\cdot f$ for every pair of $f,\, g \in G$. A familiar example of an Abelian group is the set of rotations in two dimensions. In contrast, the set of rotations in three dimensions is non-Abelian.

For the most part, we will be interested in symmetry transformations that act linearly on quantum fields,

$$\phi_i \to \phi'_i = U_{ij}\phi_j \ , \tag{22}$$

where U_{ij} is independent of the fields. As a result, we will usually work with matrix *representations* of groups. Groups themselves are abstract mathematical objects. A representation of a group is a set of $n \times n$ matrices $U(g)$, one for each group element, such that:

[1] Much of this discussion is based on Refs. [3, 2], both of which provide a much more detailed account of the topics covered here.

1. $U(f)U(g) = U(f \cdot g)$

2. $U(1) = \mathbb{I}$, the identity matrix.

Note that these conditions imply that $U(f^{-1}) = U^{-1}(f)$. The value of n is called the dimension of the representation. For any group, there is always the *trivial* representation where $U(g) = \mathbb{I}$ for every $f \in G$. Note that a representation does not have to faithfully reproduce the full multiplication table. A representation is said to be *unitary* if all the representation matrices can be taken to be unitary ($U^\dagger = U^{-1}$).

e.g. 7. Rotations in two dimensions

This group is formally called $SO(2)$ and can be defined as an abstract mathematical object. Any group element can be associated with a rotation angle θ. The most familiar representation is in terms of 2×2 matrices,

$$D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{23}$$

Of course, there is also the trivial representation, $D(\theta) = 1$.

Our focus will be primarily on continuous transformations. These correspond to what are called *Lie groups*, which are simply groups whose elements can be parametrized in terms of a set of continuous variables $\{\alpha^a\}$, where a labels the set of coordinates needed. We can (and will) always choose these coordinates (near the identity) such that the point $\alpha^a = 0$ corresponds to the identity element of the group. Thus, for any representation of a Lie group, we have for infinitesimal transformations near the identity

$$U(\alpha^a) = \mathbb{I} + i\alpha^a t^a + \mathcal{O}(\alpha^2). \tag{24}$$

The matrices t^a are called *generators* of the representation. Finite transformations can be built up from infinitesimal ones according to

$$U(\alpha^a) = \lim_{p\to\infty} (1 + i\alpha^a t^a/p)^p = e^{i\alpha^a t^a}. \tag{25}$$

This is nice because it implies that we only need to sort out a finite set of generators when discussing the representation of a Lie group rather than the infinite number of group elements.

A set of generator matrices $\{t^a\}$ can represent a Lie group provided they satisfy a *Lie algebra*. Besides being able to add and multiply them, they must also satisfy the following conditions:

1. $[t^a, t^b] = i f^{abc} t^c$ for some real constants f^{abc}

2. $[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0$ (Jacobi Identity)

The first condition is needed for the closure of the group (*i.e.* $\exp(i\alpha^a t^a)\exp(i\beta^a t^a) = \exp(i\lambda^a t^a)$ for some λ^a) while the second is required for associativity. In fact, we can define a Lie group abstractly by specifying the *structure constants* f^{abc}. Most of the representations we will work with are unitary, in which case the structure constants are all real and the generators t^a are Hermitian.

The great thing about working with linear generators t^a is that we can choose a nice basis for them. This is equivalent to choosing a nice set of coordinates for the Lie group. In particular, it is always possible to choose the generators t^a_r of any representation r such that

$$tr(t^a_r t^b_r) = T_2(r)\delta^{ab}. \tag{26}$$

The constant $T_2(r)$ is called the Dynkin index of the representation. We will always implictly work in bases satisfying Eq. (26), and we will concentrate on the case where the index is strictly positive. If so, the corresponding Lie group is said to be *compact* and is guaranteed to have finite-dimensional unitary representations. (A familiar non-compact example is the Minkowski group.)

There are only a few classes of compact Lie groups. The *classical* groups are:

- $U(1)$ = phase transformations, $U = e^{i\alpha}$
- $SU(N)$ = set of $N \times N$ unitary matrices with $det(U) = 1$
- $SO(N)$ = set of orthogonal $N \times N$ matrices with $det(U) = 1$
- $Sp(2N)$ = set of $2N \times 2N$ matrices that preserve a slightly funny inner product.

In addition to these, there are the *exceptional* Lie groups: E_6, E_7, E_8, F_4, G_2. In studying the Standard Model, we will focus primarily on $U(1)$ and $SU(N)$ groups.

e.g. 8. $SU(2)$

This is the prototypical Lie group, and should already be familiar from what you know about spin in quantum mechanics. By definition, the corresponding Lie algebra has three basis elements which satisfy

$$[t^a, t^b] = i\epsilon^{abc}t^c \tag{27}$$

The basic *fundamental* representation of $SU(2)$ is in terms of Pauli matrices: $t^a = \sigma^a/2$. Since $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, it's clear that this is a valid representation of the algebra. You might also recall that any $SU(2)$ matrix can be written in the form $U = \exp(i\alpha^a\sigma^a/2)$. Note as well that if we interpret the $SU(2)$ as spin, the fundamental representation corresponds to spin $s = 1/2$. Other spins correspond to different representations of $SU(2)$.

Some useful and fun facts about compact Lie algebras:

- Except for $U(1)$, we have $tr(t^a) = 0$ for all the classical and exceptional Lie groups.
- Number of generators $= d(G)$

$$d(G) = \begin{cases} N^2 - 1; & SU(N) \\ N(N-1)/2; & SO(N) \\ 2N(2N+1)/2; & Sp(2N) \end{cases} \tag{28}$$

- A representation (= rep) is *irreducible* if it cannot be decomposed into a set of smaller reps. This is true if and only if it is impossible to simultaneously block-diagonalize all the generators of the rep. Irreducible representation = irrep.
- If one of the generators commutes with all the others, it generates a $U(1)$ subgroup called an Abelian factor: $G = G' \times U(1)$.
- If the algebra cannot be split into sets of mutually commuting generators it is said to be *simple*. For example, $SU(5)$ is simple (as are all the classical and exceptional Lie groups given above) while $SU(3) \times SU(2) \times U(1)$ is not simple. In the latter case, all the $SU(3)$ generators commute with all the $SU(2)$ generators and so on.
- A group is *semi-simple* if it does not have any Abelian factors.
- With the basis choice yielding Eq. (26), one can show that the structure constants are completely anti-symmetric.
- The *fundamental* representation of $SU(N)$ is the set of $N \times N$ special unitary matrices acting on a complex vector space. This is often called the **N** representation. Similarly, the fundamental representation of $SO(N)$ is the set of $N \times N$ special orthogonal matrices acting on a real vector space.
- The *adjoint* (= A) representation can be defined in terms of the structure constants according to

$$(t^a_A)_{bc} = -if^{abc} \tag{29}$$

  Note that on the left side, a labels the adjoint generator while b and c label its matrix indices.

- Given any rep t^a_r, the conjugate matrices $-(t^a_r)^*$ give another representation, unsurprisingly called the conjugate representation. A rep is said to be real if it is unitarily equivalent to its conjugate. The adjoint rep is always real.
- The Casimir operator of a rep is defined by $T^2_r = t^a_r t^a_r$ (with an implicit sum on a). One can show that T^2_r commutes with all the t^a_r. For an irrep (=irreducible representation) of a simple group, this implies that

$$T^2_r = C_2(r)\mathbb{I}, \tag{30}$$

  for some positive constant $C_2(r)$.

- It is conventional to fix the normalization of the fundamental of $SU(N)$ such that $T_2(\mathbf{N}) = 1/2$. Once this is done, it fixes the normalization of all the other irreps. In particular, it implies that for $SU(N)$, $C_2(\mathbf{N}) = (N^2-1)/2N$, $T_2(A) = N = C_2(A)$.

3 Representations of Lie Groups on Quantum Fields

Let's now apply this mathematical technology to QFTs. In most of the cases we will study in this course, symmetries act as linear transformations on the field variables. For continuous symmetries, this means that sets fields in the theory will transform under representations of the symmetry group. While this might seem very restrictive, our freedom to change field variables means that it actually applies very broadly. For now, we will also concentrate on symmetries that act only on the field variables and not spacetime. Transformations that act on both will be discussed in the next section.

Consider a set of fields $\{\psi_i\}$ that transform under a representation r of some compact Lie group G. By definition this implies that under transformation by a group element with coordinates α^a the fields change as

$$\psi_i(x) \to \psi_i'(x) = \left(e^{i\alpha^a t_r^a}\right)_{ij} \psi_j(x) \ , \tag{31}$$

where the t_r^a are Hermitian generators of the representation. Specializing to infinitesimal transformations, we have

$$\psi_i \to \psi_i' = \psi_i + \alpha^a \Delta_i^a(\psi) \ , \tag{32}$$

with

$$\Delta_i^a(\psi) = i(t_r^a)_{ij}\psi_j \ . \tag{33}$$

Thus, the generators of the representation dictate the leading shifts in the fields. Note as well that we have generalized our previous notation of Eq. (3) to include a group index a to allow us to consider all the group transformations at once.

e.g. 9. Complex scalars with $SU(2)$ symmetry

An example of a QFT with a non-trivial non-Abelian symmetry is a pair of complex scalars ϕ_1 and ϕ_2 that transform into each other under the $d = 2$ irrep of $SU(2)$. Explicitly, this means

$$\phi_i(x) \to \phi_i'(x) = \left(e^{i\alpha^a \sigma^a/2}\right)_{ij} \phi_j(x) \ . \tag{34}$$

Assembling $\phi = (\phi_1, \phi_2)^t$ into a column vector, it is straightforward to write a sensible Lagrangian that is invariant under $SU(2)$:

$$\mathscr{L} = \eta^{\mu\nu}(\partial_\mu\phi)^\dagger\partial_\nu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{2}\left(\phi^\dagger\phi\right)^2 \ . \tag{35}$$

The invariance of this Lagrangian follows from $U(\alpha^a) \equiv \exp(i\alpha^a\sigma^a/2)$ being a unitary matrix for any real, constant α^a. We say that the theory has a $SU(2)$ global symmetry. It is straightforward to generalize this structure to other group representations.

4 Representations of the Poincaré Group (and More)

A really important continuous (Lie) symmetry group of relativistic QFTs is Poincaré, the combined group of spacetime translations and Lorentz transformations. This symmetry dictates the kinds of fields we consider and even the structure of quantum field theory itself. Furthermore, the quantum mechanical property of spin of elementary particles can be understood as a consequence of this invariance. We will only give a rough sketch of how this works here, but more details can be found in Refs. [6, 7]. Along the way, we will also discuss how to handle transformations that act on both the fields and spacetime itself.

The Poincaré group is a Lie group with ten generators: four spacetime translations, three rotations, and three Lorentz boosts. This symmetry is usually incorporated into QFTs by using fields that transform under definite representations of Poincaré. In contrast to the Lie groups discussed above, the Poincaré group is not compact, and its representations are typically not unitary. The spacetime translation part of the group is handled by using fields that are functions of spacetime, and correspond to what is called a *coordinate representation* of the subgroup. Expanding to also include the Lorentz portion of the group, the simplest representations on fields are the scalar ($s = 0$), Weyl fermion ($s = 1/2'$, and vector ($s = 1'$) that we have already seen.

In addition to using field operators that transform as definite representations of Poincaré, we also want to use state vectors in the quantum Hilbert space that are representations of the group. This is a bit more complicated for states because the representations in this case have to be unitary. A key feature is that they can be characterized by the Lorentz invariant P^2. For massive states, with $P^2 = M^2 > 0$, these representations are labelled unambiguously by their total momentum and spin, with the spin part corresponding to the familiar $SU(2)$ group in the particle rest frame. For massless states, $P^2 = 0$, the spin part corresponds instead to the group $ISO(2)$, and these states are characterized by their helicity, which can only take two values corresponding to spin parallel or anti-parallel to direction.

Going back to actions and Lagrangians, let's examine how to implement transformations that act on spacetime as well as on fields. Suppose we have a set of fields that transform under some representation of a continuous group with parameters α^a and corresponding matrices $M(\alpha^a)$ such that the spacetime coordinates also transform:

$$x \;\to\; x' = f(x;\alpha^a) \tag{36}$$

$$\phi(x) \;\to\; \phi'(x') = M(\alpha^a)\,\phi(x) = M(\alpha^a)\,\phi(f^{-1}(x';\alpha^a)) \;, \tag{37}$$

where $x' = f(x;\alpha^a)$ defines some transformation on the spacetime coordinates.[2] In terms of the action, we have

$$S[\phi] = \int d^4x\;\mathscr{L}(\phi(x);\partial_\mu) \;\to\; \int d^4x'\;\mathscr{L}(\phi'(x');\partial'_\mu) \;, \tag{38}$$

where for clarity we have also indicated which derivatives are to be used in the action ($\partial_\mu = \partial/\partial x^\mu$, $\partial'_\mu = \partial/\partial x'^\mu$).

[2] Recall that we wrote this down for the special case of Lorentz transformations in Eqs. (2,3) in `notes-01`.

***e.g. 10.* Translations**

A key illustration of all this is the subgroup of Poincaré of spacetime translations,

$$x^\lambda \to x^\lambda - a^\lambda \ . \tag{39}$$

This subgroup is Abelian because the order of multiple translations doesn't matter. To see what it implies, consider a QFT with a single real scalar field with action

$$\mathscr{L} = (\partial\phi)^2/2 - V(\phi) \ . \tag{40}$$

Under an arbitrary translation we have

$$\phi(x) \to \phi'(x') = \phi(x) = \phi(x'+a) \ , \tag{41}$$

and in the action

$$\begin{aligned} S[\phi] \ \to \ S[\phi'] \ &= \ \int d^4x' \, \frac{1}{2}(\partial'\phi'(x'))^2 - V(\phi'(x')) \\ &= \ \int d^4x \, \frac{1}{2}(\partial\phi(x))^2 - V(\phi(x)) \ = \ S[\phi] \ , \end{aligned} \tag{42}$$

where the second equality follows from $\phi'(x') = \phi(x)$, $d^4x' = d^4x$, and $\partial'_\mu = \partial_\mu$. Note that even though we've illustrated this for a simple scalar theory, the same arguments apply to any action in which all the spacetime dependence comes from dynamical fields. Put another way, by using quantum fields and actions defined on all spacetime, we automatically incorporate a symmetry under spacetime translations!

Next, let's extract the Noether currents for these four translation symmetries. Expanding for infinitesimal translations, we have

$$\phi'(x) = \phi(x+a) \ \simeq \ \phi(x) + a^\lambda \partial_\lambda \phi \ , \tag{43}$$

and thus $\Delta_\lambda(\phi) = \partial_\lambda\phi$. Applying this to the Lagrangian, we find

$$\mathscr{L}(\phi') = \mathscr{L}(\phi) + a^\lambda \partial_\mu \left(\delta^\mu_\lambda \mathscr{L}\right) \ , \tag{44}$$

so spacetime translations are a symmetry of our theory with $K^\mu_\lambda = \delta^\mu_\lambda \mathscr{L}$. The corresponding conserved currents are

$$j^\mu_\lambda = \partial^\mu\phi \, \partial_\lambda\phi - \delta^\mu_\lambda \mathscr{L} \ . \tag{45}$$

At this point, let us emphasize that we have just considered four different symmetries at once; each value of $\lambda = 0, 1, 2, 3$ corresponds to a different transformation.[3] In contrast, μ labels the spacetime index that always arises on the current. However, since $\eta_{\mu\nu} j^\nu_\lambda = j_{\mu\lambda} = j_{\lambda\mu}$ in this case, we can afford to be a bit careless with the indices.

For time translations specifically, we should take $\lambda = 0$. The corresponding charge is

$$\int d^3x \ j^0_0 = \int d^3x \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right] \ . \tag{46}$$

[3]This is why we used λ instead of ν.

This is the Hamiltonian H of the system, and thus invariance under time translations corresponds to energy conservation, $\dot{H} = 0$. Similarly, for spatial translations the related charge is

$$\int d^3x\, j_i^0 = \int d^3x\, (\partial_t\phi)\, \partial_i\phi \ , \tag{47}$$

corresponding to a conserved spatial momentum P_i. Given the important physical interpretation of j_λ^μ, it is traditionally assigned a special symbol:

$$j^{\mu\nu} \equiv T^{\mu\nu} \ , \tag{48}$$

and is called the *energy-momentum tensor*. The related charges are usually combined into a single conserved 4-vector, $P^\mu = \int d^3x\, j^{0\mu} = (H, \vec{P})$.

e.g. 11. **Scale transformations**

Scale transformations (or *dilatations*) correspond to stretching spacetime uniformly. Specifically,

$$x \ \to \ x' = \xi^{-1}x \ , \tag{49}$$

for some non-zero real number ξ. Consider applying this transformation to the theory

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - \frac{\lambda}{4!}\phi^4 - y\,\phi\,\bar{\psi}\psi \ , \tag{50}$$

where we also take

$$\phi(x) \ \to \ \phi'(x') = \xi\,\phi(x) \tag{51}$$

$$\psi(x) \ \to \ \psi'(x') = \xi^{3/2}\psi(x) \ . \tag{52}$$

In the action, we find

$$\begin{aligned} S[\phi,\psi] \ &\to \ \int d^4x' \left[\frac{1}{2}(\xi\partial'\phi)^2 + \xi^3\bar{\psi}'i\gamma^\mu\partial'_\mu\psi - \xi^4\,\frac{\lambda}{4!}\phi^4 - \xi^4\,y\,\phi\,\bar{\psi}\psi\right] \\ &= \ \int \left(d^4x\,\xi^{-4}\right)\xi^4\left[\frac{1}{2}(\partial\phi)^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - \frac{\lambda}{4!}\phi^4 - y\,\phi\,\bar{\psi}\psi\right] \\ &= \ S[\phi,\psi] \ , \end{aligned} \tag{53}$$

where we have used $d^4x' = d^4x\,\xi^{-4}$ and $\partial'_\mu = \xi\partial_\mu$. We seethat the factors of ξ cancel out completely, and scale transformations are a symmetry of the action.

There a couple of neat things to point out here as well as word of warning. First, doing dimensional analysis it is easy to check that the rescaling of all the relevant quantities here corresponds to their mass dimension (in natural units). This is key to the invariance and not an accident. Second, and related, all the parameters in the Lagrangian (λ and y) are dimensionless. It is straightforward to check that a Lagrangian with dimensionful parameters (such as mass terms) would not be invariant. While this is all great, our warning is that while the symmetry here works for the classical theory, it is typically broken by quantum effects in the full quantum theory. This is an example of something called an *anomaly*, and we will discuss them more later in the course.

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