P528 Notes #2: Calculating Stuff

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We turn next to the serious business of calculating cross sections and decay rates. The underlying material needed for this was covered in the previous notes. Here we go over the more mechanical (but essential) aspects of extracting theoretical predictions that can be compared with experiment.

For a given theory, the general strategy is to identify the physical excitations, derive the Feynman rules, and apply them to obtain the matrix element $\mathcal{M}$ for the processes of interest. Squaring the matrix element and putting it into the cross section (or decay) formulas of `notes-00` (which we reproduce here in Appendix 3) then allows us to find the corresponding physical observables. The best way to see how this works is through explicit examples, and this is how we proceed here.

Before getting to these examples, let us mention an important delta function identity that we will use frequently in evaluating cross sections. For any nice functions $f(x)$ and $g(x)$ we have

$$\int_a^b dx\;\delta\Big(f(x)\Big)\,g(x) = \sum_i \int_a^b dx\;\delta(x-x_i)\left|\frac{1}{f'(x)}\right|\,g(x)\;, \tag{1}$$

where the sum runs over all values x_i for which $f(x_i) = 0$.

1 Example: $\phi\phi \to \phi\phi$ in $\lambda\,\phi^4$ Theory

Consider the process $\phi(p_1)+\phi(p_2) \to \phi(p_3)+\phi(p_4)$ in the $\lambda\phi^4$ theory discussed in `notes-01`. Recall that the scattering matrix element at leading order in the coupling λ was just

$$-i\,\mathcal{M} = -i\,\lambda\;. \tag{2}$$

It is independent of the initial and final momenta and there are no spins to deal with. For scattering in the centre-of-mass (CM) frame, defined by $\vec{p}_1+\vec{p}_2 = \vec{0}$, we can align the z-axis with the initial p_1 direction so that the initial 4-momenta take the form

$$p_1 = (E,0,0,p)\;,\quad p_2 = (E,0,0,-p)\;, \tag{3}$$

where $E = \sqrt{m^2+p^2}$.

To compute the scattering cross section, we apply the formula of Eq. (24) collected in the appendix. Before trying to get the final result, it is useful to assemble the pieces that go into it. The symmetry factor in this formula is $S = 1/2!$ since there are two identical particles in the final state. The relative velocity is

$$\begin{aligned} v &= \sqrt{(p_1\cdot p_2)^2 - m^4}\Big/E^2 \\ &= 2\,p/E\;. \end{aligned} \tag{4}$$

This goes to $v \to 2$ for $p \gg m$ and $v \simeq 2\,p/m$ for $p \ll m$. Moving next to the integrals, it turns out that most of them can be done using the overall 4-momentum delta function. Explicitly,

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\; \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\ . \tag{5}$$

Thus, the 3-momentum part of the full delta function can be used to do one of the d^3p_i integrals. Let us use it to eliminate the d^3p_4 integral. Doing so, we must set

$$\vec{p}_4 \;=\; \vec{p}_1 + \vec{p}_2 - \vec{p}_3 \;=\; -\,\vec{p}_3$$

where in the second equality we have used the fact that we are working in the CM frame. Since the two outgoing masses are equal, this also implies that $E_3 = E_4$. For the remaining d^3p_3 integral, we can decompose it spherically into $d^3p_3 = d\Omega_3\ dp'p'^2$, where $p' = |\vec{p}_3|$ is the magnitude and $d\Omega_3$ runs over the spherical direction of $\vec{p}_3$. The remaining energy delta function allows us to do the dp' integral. Specifically, it takes the form

$$\begin{aligned}\int_0^\infty dp'\ \delta\Big(2E - 2\sqrt{m^2+p'^2}\Big)\,\mathcal{F}(p') \;&=\; \int_0^\infty dp'\ \delta(p'-p)\,\left|\frac{d}{dp'}\Big(2\sqrt{m^2+p'^2}\Big)\right|^{-1} \mathcal{F}(p')\\ &=\; \frac{\sqrt{m^2+p^2}}{2p}\ \mathcal{F}(p)\ ,\end{aligned} \tag{6}$$

where $\mathcal{F}(p')$ stands for the p' dependence of the rest of the integrand. In the first line we used the delta function identity of Eq. (1) together with $p' = p$ being the only positive solution of $\ 2E - 2\sqrt{m^2+p'^2} = 0$.

Putting all these pieces together, the cross section is

$$\begin{aligned}\sigma \;&=\; \frac{1}{2v}\frac{1}{4E^2}\frac{1}{(2\pi)^2}\int\frac{d^3p_3}{2E_3}\int\frac{d^3p_4}{2E_4}\ \delta^{(4)}(p_1+p_2-p_3-p_4)\,|\mathcal{M}|^2 &(7)\\ &=\; \frac{1}{32\,v}\ \frac{1}{(2\pi)^2}\ \frac{1}{E^2}\ \int d\Omega_3\ \frac{1}{E^2}\frac{E}{2p}\,p^2\,|\mathcal{M}|^2 &(8)\\ &=\; \frac{1}{v}\ \frac{1}{64}\ \frac{1}{(2\pi)^2}\ \frac{1}{E^2}\ \frac{p}{E}\ \int d\Omega_3\ |\mathcal{M}|^2 &(9)\\ &=\; \frac{\lambda^2}{128\pi}\frac{1}{E^2}\ . &(10)\end{aligned}$$

In the last line we have used $|\mathcal{M}|^2 = \lambda^2$, $v = 2p/E$, and $\int d\Omega_3 = 4\pi$. More generally, the matrix element will have an angular dependence and the integral over solid angle will not be trivial. Note as well that $d\Omega_3$ runs over angles relative to the initial ϕ direction (which we chose to be the z-axis). Thus, the *differential cross section* $d\sigma/d\Omega = d^2\sigma/[d(\cos\theta)d\phi]$ is given by the expression of Eq. (9) above but with the integral over Ω_3 left out.

2 Example: $e^-e^+ \to \mu^-\mu^+$ in QED

We computed the matrix element for $e^-(p_1,s_1) + e^+(p_2,s_2) \to \mu^-(p_3,s_3) + \mu^+(p_4,s_4)$ in `notes-01`, where $\{p_i, s_i\}$ label the momentum and spin state of the i-th particle. At leading

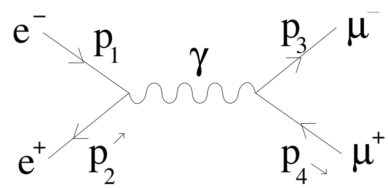


Figure 1: Leading Feynman diagram for $e^+e^- \to \mu^+\mu^-$.

order in the electromagnetic coupling there is the single Feynman diagram shown in Fig. 1. To obtain the matrix element, a standard approach is to track backwards along fermion lines and collect the various fermionic and vertex factors along the way. Once this has been done for all fermion lines, the propagators and external spin factors for all other lines in the diagram can be added. Doing so here yields

$$-i\mathcal{M} \;=\; [\bar{u}_3\,(ie\gamma^\mu)\,v_4]\,[\bar{v}_2\,(ie\gamma^\nu)\,u_1]\left(\frac{-i\,\eta_{\mu\nu}}{q^2}\right)\;, \tag{11}$$

where $q = (p_1 + p_2)$, u_i and v_i are the external spin polarizations of the i-th fermion (with momentum p_i and spin stata s_i), and we have set $Q_e = Q_\mu = -1$. Note that each term in square brackets is a scalar in Dirac space, having the form of row vector times matrix times column vector. These two terms are connected (in their Lorentz indices) by the photon propagator.

To find the cross section for this process, we need to specify the fermion spin polarizations, compute the result, and square it within the cross section formula. The result would be the cross section for electrons and positrons with specific initial spin states to scatter into muons and antimuons with specific final spin states. However, in most experiments the incident beams are typically unpolarized (having collections of uniformly random spins) and the spins of the final state particles are not measured. To determine the effective total cross section in this type of scenario, we should *average* over initial spin states and *sum* over final spin states. It turns out that there are a number of simplifying tricks for doing this that make use of the completeness properties of spin polarizations. We collect the most important of these for Dirac spinors and photon polarizations in Appendix 3.

Returning to the specific problem at hand, we want

$$\begin{aligned}
\text{“}|\mathcal{M}|^{2\prime\prime} \;&\equiv\; \frac{1}{2}\times\frac{1}{2}\times\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}(s_1s_2\to s_3s_4)|^2 \qquad (12)\\
&=\; \frac{1}{4}\left(\frac{e^2}{q^2}\right)^2\sum_{\{s_i\}}\eta_{\mu\nu}(\bar{u}_3\gamma^\mu v_4)(\bar{v}_2\gamma^\nu u_1)[\eta_{\alpha\beta}(\bar{u}_3\gamma^\alpha v_4)(\bar{v}_2\gamma^\beta u_1)]^*\;.
\end{aligned}$$

Let us first conjugate the $\{34\}$ spinor piece. We have

$$\begin{aligned}
(\bar{u}_3\gamma^\alpha v_4)^* \;=\; (\bar{u}_3\gamma^\alpha v_4)^\dagger \;&=\; v_4^\dagger(\gamma^\alpha)^\dagger\gamma^0(u_3^\dagger)^\dagger \qquad (13)\\
&=\; v_4^\dagger\gamma^0\gamma^0(\gamma^\alpha)^\dagger\gamma^0 u_3\\
&=\; \bar{v}_4\gamma^\alpha u_3\;.
\end{aligned}$$

The {12} piece goes through similarly. Next, we assemble the {12} and {34} pieces and use the spinor completeness relations. For the {34} part, we get

$$\begin{aligned}\sum_{s_3,s_4}(\bar{u}_3\gamma^\mu v_4)\,(\bar{v}_4\gamma^\alpha u_3) &= \sum_{s_3}\sum_{s_4}\bar{u}_{3a}\,\gamma^\mu_{ab}\,v_{4b}\,\bar{v}_{4c}\,\gamma^\alpha_{cd}\,u_{3d}\\ &= (\not{p}_3+m_3)_{da}\,\gamma^\mu_{ab}\,(\not{p}_4-m_4)_{bc}\,\gamma^\alpha_{cd}\\ &= tr[(\not{p}_3+m_3)\gamma^\mu(\not{p}_4-m_4)\gamma^\alpha]\\ &= 4(p_3^\mu p_4^\alpha+p_3^\alpha p_4^\mu-p_3\!\cdot\! p_4\eta^{\mu\alpha})-4m_3m_4\eta^{\mu\alpha}\ .\end{aligned}$$

We have written out the spinor indices in gory detail here, but you can skip this once you get the hang of it. Combining with the {12} piece and contracting indices, the result is

$$\text{“}|\mathcal{M}|^2\text{''} = 8\left(\frac{e^2}{q^2}\right)^2\,[(p_1\!\cdot\! p_3)(p_2\!\cdot\! p_4)+(p_1\!\cdot\! p_4)(p_2\!\cdot\! p_3)+(\ldots)]\ , \tag{14}$$

where $(\ldots)$ refers to stuff that depends on the masses m_e and m_μ. For now, let us focus on the high-energy limit $q^2 \gg m_\mu^2,\, m_e^2$ so that the masses can be neglected. Working in the centre-of-mass (CM) frame, we have (after applying energy and momentum conservation and choosing a nice set of axes)

$$p_1 = (p,0,0,p)\ ,\qquad p_2=(p,0,0,-p) \tag{15}$$

$$p_3 = (p,p\sin\theta,0,p\cos\theta)\ ,\quad p_4=(p,-p\sin\theta,0,-p\cos\theta) \tag{16}$$

for the initial momentum p. The summed and squared matrix element is then

$$\text{“}|\mathcal{M}|^2\text{''} = e^4\,(1+\cos^2\theta)\ . \tag{17}$$

This can be used to compute the unpolarized cross section.

3 Example: $e^-\gamma \to e^-\gamma$ in QED

This process is usually called Compton scattering. The full differential cross section at leading order in α is given by the *Klein-Nishina formula*, which reduces to the *Thomson cross section* at low-energies in the electron rest frame. Since the full calculation is a bit tedious, we will only present some highlights of it, with the main goal being to illustrate how to handle external photons.

Labelling the momenta by $e^-(p_1)+\gamma(p_2)\ \to\ e^-(p_3)+\gamma(p_4)$, there are two Feynman diagrams for the process at leading order in the electromagnetic coupling, shown in Fig. 2. The full amplitude is the sum of them, and can be written in the form

$$-i\mathcal{M} = -i\,(\mathcal{M}_a^{\mu\nu}+\mathcal{M}_b^{\mu\nu})\,\epsilon_\mu^*(p_4)\,\epsilon_\nu(p_2)\ , \tag{18}$$

where $\epsilon_\mu(p_i)$ is a photon polarization vector and

$$-i\mathcal{M}_a^{\mu\nu} = \bar{u}_3(ie\gamma^\mu)\frac{i(\not{q}+m)}{q^2-m^2}(ie\gamma^\nu)u_1 \tag{19}$$

$$-i\mathcal{M}_b^{\mu\nu} = \bar{u}_3(ie\gamma^\nu)\frac{i(\not{k}+m)}{k^2-m^2}(ie\gamma^\mu)u_1\ , \tag{20}$$

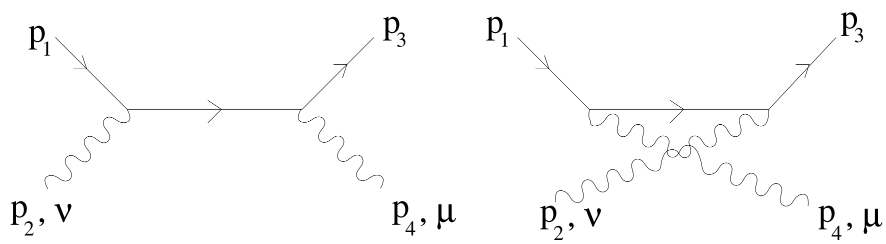


Figure 2: Leading Feynman diagrams for Compton scattering, $e^-\gamma \to e^-\gamma$.

with $q = (p_1+p_2)$ and $k = (p_1-p_4)$. Note that both amplitudes add together constructively, but have different orderings of Dirac matrices and internal momenta. To get the full unpolarized summed and squared matrix element, we follow the usual procedure and average over initial states and sum over final ones. This gives[1]

$$\begin{aligned}\text{“}|\mathcal{M}|^2\text{”} &= \frac{1}{2}\times\frac{1}{2}\sum_{\lambda_2,\lambda_4}\sum_{s_1,s_3}(\mathcal{M}_a^{\mu\nu}+\mathcal{M}_b^{\mu\nu})\,\epsilon_\mu^*(p_4)\,\epsilon_\nu(p_2)\left(\mathcal{M}_a^{*\,\alpha\beta}+\mathcal{M}_b^{*\,\alpha\beta}\right)\epsilon_\alpha(p_4)\,\epsilon_\beta^*(p_2) \\ &= \frac{1}{4}\sum_{\lambda_2}\epsilon_\nu(p_2)\,\epsilon_\beta^*(p_2)\sum_{\lambda_4}\epsilon_\mu^*(p_4)\,\epsilon_\alpha(p_4)\sum_{s_1,s_3}(\mathcal{M}_a^{\mu\nu}+\mathcal{M}_b^{\mu\nu})\left(\mathcal{M}_a^{*\,\alpha\beta}+\mathcal{M}_b^{*\,\alpha\beta}\right)\ .\end{aligned} \tag{21}$$

The sums on λ_2 and λ_4 here run over the possible polarization states of the initial and final photons, while the sums on s_1 and s_3 run over the electron spin states. Note that both the initial electron and photon each have two independent states which produce an overall factor of $1/4 = 1/2\times 1/2$ in the average. To handle the polarization sums, we use the photon trick listed in Appendix 3 to give

$$\sum_{\lambda_2}\epsilon_\nu^*(p_2)\,\epsilon_\beta(p_2) = -\eta_{\nu\beta}\ , \tag{22}$$

with a corresponding result of $(-\eta_{\mu\alpha})$ for the λ_4 sum. The fermion pieces are more involved and require a good deal of Dirac algebra. For example, the (aa) term gives

$$\mathcal{M}_a^{\mu\nu}\mathcal{M}_a^{*\,\alpha\beta} = \left(\frac{e^2}{q^2-m^2}\right)^2 tr\left[(\not{p}_3+m)\gamma^\mu(\not{q}+m)\gamma^\nu(\not{p}_1+m)\gamma^\beta(\not{q}+m)\gamma^\alpha\right] \tag{23}$$

The trace can be done by brute force, but it can also be simplified using the Dirac equation and anticommutation relations. This is left to the intrepid reader.

[1]When squaring, make sure to use different dummy indices in the two factors to avoid mixing them up!

Appendix A: Cross Section and Decay Rate Formulas

For $2 \to n$ scattering with two initial particles colliding to make a final state with n particles, let us label the initial 4-momenta by p_1 and p_2 and the final 4-momenta by $p_3, \ldots, p_{n+2}$. The formula for the scattering cross-section is

$$\sigma = \frac{S}{v} \frac{1}{4E_1 E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \cdots \int \frac{d^3 p_{n+2}}{(2\pi)^3 2E_{n+2}} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_{i=3}^{n+2} p_i) \, |\mathcal{M}|^2 \ , \tag{24}$$

where $v = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \Big/ E_1 E_2$ is the magnitude of the relative velocity of the incident particles, and S is a combinatoric factor equal to one times $1/k!$ for every set of k identical particles in the final state. Derivations of this result can be found in the textbooks by Peskin & Schroeder [2] and Srednicki [4].

The formula for the partial decay rate of an unstable particle of mass M at rest to decay to a final state containing n particles ($M \to 1 + 2 + \ldots + n$) is [2]

$$\Gamma(M \to n) = \frac{S}{2M} \int \frac{d^3 p_1}{2E_1 (2\pi)^3} \cdots \int \frac{d^3 p_n}{2E_n (2\pi)^3} (2\pi)^4 \delta^{(4)}(p_M - \sum_{i=1}^{n} p_i) \, |\mathcal{M}|^2 \ , \tag{25}$$

where $|\mathcal{M}|^2$ is the corresponding $M \to n$ amplitude defined in the same way as for scattering, and S is the symmetry factor. The total decay rate is the sum of all the partial decay widths,

$$\Gamma = \sum_f \Gamma_f \ , \tag{26}$$

where the sum runs over all possible final states. In natural units, the lifetime of the unstable particle in its rest frame is $\tau = 1/\Gamma$.

Appendix B: Dirac (and Pauli) Matrices

It is useful to generalize the 2×2 Pauli matrices to

$$\sigma^0 = \mathbb{I}, \quad \sigma^i = \sigma^{1,2,3} \tag{27}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{28}$$

Recall that

$$\sigma^i \sigma^j = \delta^{ij} \mathbb{I} + i \epsilon^{ijk} \sigma^k \ . \tag{29}$$

Let us also define

$$\sigma^\mu = (\mathbb{I}, \vec{\sigma}) \ , \qquad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma}) \ . \tag{30}$$

In terms of these, the 4×4 Dirac matrices in the so-called *chiral representation* are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{31}$$

They satisfy the familiar relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{32}$$

We also define

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\lambda\kappa}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma\kappa . \tag{33}$$

In the chiral representation, one finds

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{34}$$

We will also encounter the chiral projectors $P_L = (1 - \gamma^5)/2$ and $P_R = (1 + \gamma^5/2)$.

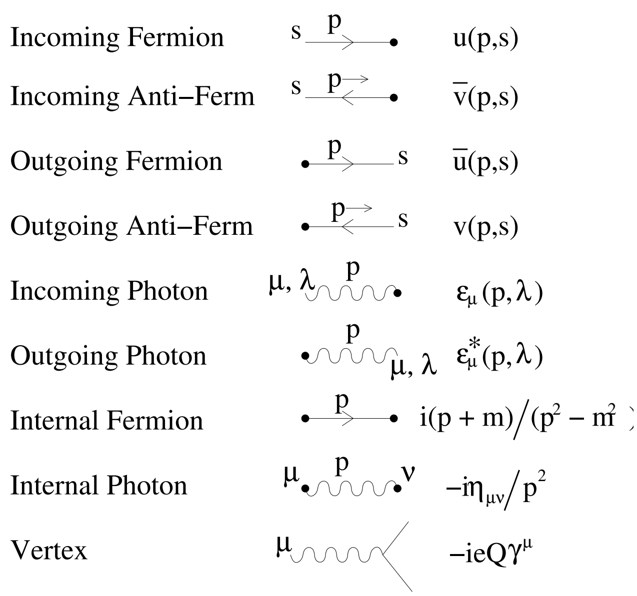
Appendix C: QED Feynman Rules and Spin Tricks

QED is the theory of charged fermions interacting with the photon. It consists of a massless vector for the photon and a set of charged fermions. The Lagrangian is [2]

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_i \bar{\psi}_i[i\gamma^\mu(\partial_\mu + ieQ_iA_\mu) - m_i]\psi_i \tag{35}$$

Sometimes we will write $D_\mu = (\partial_\mu + ieQ_iA_\mu)$, which is called a *covariant derivative.*

From this Lagrangian one can derive the following Feynman rules:



Here, $u(p,s)$ and $v(p,s)$ are 4×1 fermion and anti-fermion spin vectors for 4-momentum p and spin state s, with $\bar{u} = u^\dagger\gamma^0$. We have also written $\not{p} = p_\mu\gamma^\mu$. For photons, $\epsilon_\mu(p,\lambda)$ is the polarization 4-vector for the polarization state λ. Recall that photons have two independent transverse polarizations.

Spinor Tricks:

$$
\begin{aligned}
(\not{p} - m)\, u(p,s) &= 0 = (\not{p} + m)\, v(p,s) && (36)\\
\sum_s u_a(p,s)\bar{u}_b(p,s) &= (\not{p} + m)_{ab} && (37)\\
\sum_s v_a(p,s)\bar{v}_b(p,s) &= (\not{p} - m)_{ab} && (38)\\
\gamma^0(\gamma^\mu)^\dagger\gamma^0 &= \gamma^\mu && (39)\\
tr(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} && (40)\\
tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa) &= 4(\eta^{\mu\nu}\eta^{\lambda\kappa} + \eta^{\mu\kappa}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\kappa}) && (41)\\
tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa\gamma^5) &= -4i\epsilon^{\mu\nu\lambda\kappa} && (42)\\
\{\gamma^\mu\gamma^\nu\} &= 2\eta^{\mu\nu} && (43)\\
\{\gamma^5, \gamma^\mu\} &= 0 && (44)
\end{aligned}
$$

Additional tricks can be found in Ref. [2]. Note that the subscripts in Eqs.(37,38) are Dirac spinor indices.

Photon Tricks:

$$
\begin{aligned}
\sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda) &= -\eta_{\mu\nu} + (\text{stuff you can ignore}) && (45)\\
p^\mu\epsilon_\mu(p,\lambda) &= 0 && (46)
\end{aligned}
$$

References

[1] For an informal and very intuitive introduction to QFT, take a look at:
A. Zee, "Quantum field theory in a nutshell," *Princeton, UK: Princeton Univ. Pr. (2003) 518 p*

[2] This is the standard modern QFT textbook. It is probably the most comprehensive single text on the subject with excellent exercises. If you want to learn QFT and have to choose a single book, this is it. However, because it is so comprehensive, some specific topics are treated in more detail elsewhere.
M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[3] A good complement to Peskin and Schroeder:
L. H. Ryder, "Quantum Field Theory," *Cambridge, Uk: Univ. Pr. (1985) 443p*

[4] A very nice newer QFT textbook is:
M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*

[5] An even newer and also very nice QFT textbook:
M. D. Schwartz, "Quantum Field Theory and the Standard Model," *Cambridge, UK: Univ. Pr. (2013) 863 p,* http://www.schwartzqft.com/

[6] This book gives a non-standard but very clever introduction to QFT, and some basic string theory as well. There are a number of topics in this book that can be useful but are rarely covered in other texts, such as the Schrödinger representation of QFT in terms of wave functionals:
B. Hatfield, "Quantum field theory of point particles and strings," *Redwood City, USA: Addison-Wesley (1992) 734 p. (Frontiers in physics, 75)*

[7] A link to my notes from UBC PHYS-526:
D. E. Morrissey, "PHYS-526-2013",
http://trshare.triumf.ca/ dmorri/Teaching/PHYS526-2013/

[8] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[9] D. Tong, "Lectures on Quantum Field Theory"
http://www.damtp.cam.ac.uk/user/dt281/qft.html.

[10] W. Greiner and J. Reinhardt, "Field quantization," *Berlin, Germany: Springer (1996) 440 p.*