P528 Notes #3: Symmetries

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January 25, 2021

Symmetries play a central role in modern particle physics. Insofar as we believe that elementary particles can be described by QFT (and the evidence so far points in this direction), our role as theoretical and experimental particle physicists is to figure out the Lagrangian of our world. In particular, we must specify a set of fields and their interactions. Once we have a candidate Lagrangian, we can compute the dynamics of the theory and compare to experiment. Symmetries make the task of figuring out the Lagrangian much easier because they strongly constrain the set of possible fields and interactions. They are also enormously useful in computing the dynamics because they relate different sets of solutions to the system.

In this note, we will first describe how symmetries are dealt with in QFT. Next, we will generalize the notion of symmetries to gauge invariance as it applies to QED. And finally, we will give some formal details of the mathematical description of symmetries that will be useful later on.

1 Symmetries in QFT

Let us begin with the treatment of symmetries in QFT. We will show that if a system has a continuous symmetry, there is a related conservation law. Before getting to this key result, called *Noether's theorem*, we will start with a few examples.

e.g. 1. Discrete scalar field symmetry

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ .$$

This theory is symmetric under $\phi \to -\phi$ in that the form of the Lagrangian stays the same after the transformation. The implication of this symmetry is that for any process, the number of particles in the initial state minus the number in the final state must be even. Note that $\phi \to -\phi$ would not be a symmetry of the theory if we were to add a cubic $A\phi^3$ term to the Lagrangian.

e.g. 2. Discrete chiral symmetry

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - y\phi\bar{\psi}\psi \ .$$

This theory is symmetric under the simultaneous transformations $\phi \to -\phi$ and $\psi \to \gamma^5\psi$. (Note that the second condition implies $\bar{\psi} \to -\bar{\psi}\gamma^5$.) This symmetry is only possible if there is no fermion mass term. Such symmetries are sometimes called *chiral* symmetries.

e.g. 3. Continuous field symmetry

$$\mathscr{L} = |\partial\phi|^2 - M^2|\phi|^2 + \sum_{i=1}^{2} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - (y\,\phi\bar{\psi}_1\psi_2 + h.c.)\ .$$

This theory is symmetric under $\psi_1 \to e^{i\alpha Q_1}\psi_1$, $\psi_2 \to e^{i\alpha Q_2}\psi_2$, $\phi \to e^{i\alpha Q_\phi}\phi$ for any real constant α provided $(Q_\phi - Q_1 + Q_2) = 0$. These Q's are sometimes called the *charges* of the fields under the symmetry. In contrast to the previous examples, this symmetry is *continuous* rather than *discrete* in that it holds for any value of real parameter α.

The definition of a symmetry is a transformation of the system that the leaves the physics the same. For a field theory defined by an action that depends on a set of fields, this will be the case if and only if the transformed action has the same functional form as the original action. More precisely, for an action $S[\phi]$ that depends on the set of fields $\{\phi_i\}$ and the transformation

$$\phi_i \to \phi_i' = f_i(\phi)\ , \qquad S[\phi] \to S[\phi'] := S'[\phi]\ , \tag{1}$$

we must have $S'[\phi] = S[\phi]$ for this to be a symmetry. Equivalently, the transformed Lagrangian must have the same form as the original Lagrangian up to total derivatives. Some additional discussion of this is given in Refs. [1, 2].

Continuous symmetries are especially interesting because they imply conservation laws. This relationship is called *Noether's theorem.* Before deriving it, it is worth remembering how one obtains the classical equations of motion for a system from the action that defines it. The solution to the equation of motion is the field configuration such that for any arbitrary set of infinitesimal field variations that vanish on the boundary, $\phi_i \to \phi + \delta\phi_i$, the numerical value of the action remains the same to leading order: $\delta S = 0$. For a local field theory action, this leads to

$$\delta S = \int d^4x\, \sum_i \left(\left[\frac{\partial\mathscr{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i + \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right]\right)\ . \tag{2}$$

The second term above is zero automatically because the field variation is assumed to vanish on the boundary. Thus, demanding that $\delta S = 0$ for any small variation $\delta\phi_i$ implies that the bracketed quantity in the first term vanishes. These are the equations of motion for ϕ_i.

Consider now a continuous symmetry transformation parametrized by the dimensionless real variable α. For very small $|\alpha| \ll 1$, the leading variation in field ϕ_i can be written as

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \alpha\Delta\phi_i(x)\ . \tag{3}$$

Since this is a symmetry, we must have to leading order in α

$$\mathscr{L}(\phi') = \mathscr{L}'(\phi) = \mathscr{L}(\phi) + \alpha\,\partial_\mu K^\mu\ , \tag{4}$$

for some K^μ. This is just an explicit statement that the transformed Lagrangian keeps the same form up to a possible total derivative (*i.e.* $\alpha\,\partial_\mu K^\mu$). Plugging the form of Eq. (3) into Eq. (4) and expanding to linear order, we find

$$\begin{aligned}
\alpha\partial_\mu K^\mu &= \mathscr{L}'(\phi) - \mathscr{L}(\phi) \\
&= \sum_i \left[\frac{\partial\mathscr{L}}{\partial\phi_i}\alpha\Delta\phi_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\alpha\Delta\phi_i)\right] \\
&= \alpha\sum_i \left(\left[\frac{\partial\mathscr{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right]\Delta\phi_i + \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i\right]\right) .
\end{aligned} \tag{5}$$

The first term above vanishes by the equation of motion, while the second term remains. Rearranging and removing the common factor of α, this implies

$$0 = \partial_\mu j^\mu\ , \qquad \text{where}\ \ j^\mu := \left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i - K^\mu\right] . \tag{6}$$

This result is *Noether's theorem.*

The 4-vector j^μ obtained from Noether's theorem is said to be a *conserved current.* Noether's theorem implies that there exists such a conserved current for every continuous symmetry of the system. The current is said to be conserved because if we define the conserved charge (not necessarily electric charge!) by

$$Q = \int d^3x\ j^0\ , \tag{7}$$

we find that

$$\partial_t Q \ = \ \int d^3x\ \partial_t j^0 = \int d^3x\ \vec{\nabla}\cdot\vec{j} = 0\ . \tag{8}$$

Note that we get zero because everything vanishes on the boundary, by assumption. The physical interpretation of $j^\mu = (j^0, \vec{j})$ is that j^0 is a charge density and $\vec{j}$ is a current density.

It should be mentioned that everything we have done applies to the classical field theory defined by the action. However, these results also apply to the QFT derived from the action when they are interpreted as *operator equations*, meaning that the equations of motion and conservation laws are satisfied as operator expectation values between any states (up to a few subtleties). In one-particle quantum mechanics, the corresponding relation is known as Ehrenfest's theorem.

e.g. 4. A two-field example

Consider a theory with two real fields ϕ_1 and ϕ_2:

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right) \\
&= \frac{1}{2}(\partial\phi)^t(\partial\phi) - \frac{1}{2}m^2\phi^t\phi\ ,
\end{aligned}$$

where $\phi = (\phi_1, \phi_2)^t$. This theory is symmetric under transformations of the form

$$\phi \to \phi' = \mathcal{O}\phi \ , \tag{9}$$

where $\mathcal{O}$ is any 2×2 orthogonal matrix – satisfying $\mathcal{O}^t\mathcal{O} = \mathbb{I}$. Up to a few signs, any such matrix can be parametrized in terms of the single parameter α:

$$\mathcal{O} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \ . \tag{10}$$

Applying this transformation to the Lagrangian, we find that $\mathscr{L}(\phi') = \mathscr{L}(\phi)$, and thus $K^\mu = 0$. For small rotation angles α, the transformation becomes

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 + \alpha\Delta\phi_1 \\ \phi_2 + \alpha\Delta\phi_2 \end{pmatrix} \ . \tag{11}$$

Thus, $\Delta\phi_1 = -\phi_2$ and $\Delta\phi_2 = \phi_1$. The conserved current is therefore

$$j^\mu = -(\partial^\mu\phi_1)\phi_2 + \phi_1(\partial^\mu\phi_2) \ . \tag{12}$$

It is straightforward to check that this current is indeed conserved, $\partial_\mu j^\mu = 0$.

e.g. 5. Complex scalar field

Recall that the theory of *e.g. 4.* can be rewritten in terms of a single complex scalar $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The Lagrangian becomes

$$\mathscr{L} = |\partial\phi|^2 - m^2|\phi|^2 \ . \tag{13}$$

The symmetry transformation above can now be written in the form

$$\phi \to \phi' = e^{i\alpha}\phi \ , \qquad \phi^* \to \phi'^* = e^{-i\alpha}\phi^* \ . \tag{14}$$

(Note that ϕ and ϕ^* should be thought of as independent "real" variables.) Specializing to infinitesimal α, we have

$$\Delta\phi = i\phi \ , \qquad \Delta\phi^* = -i\phi^* \ . \tag{15}$$

This gives the current

$$j_\mu = -i\,\phi^*\overset{\leftrightarrow}{\partial}_\mu\phi \tag{16}$$

where $\overset{\leftrightarrow}{\partial}_\mu = (\overset{\rightarrow}{\partial}_\mu - \overset{\leftarrow}{\partial}_\mu)$

e.g. 6. Dirac fermion

Consider

$$\mathscr{L} = \overline{\psi}i\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \ . \tag{17}$$

This theory has a symmetry under

$$\psi \to e^{iQ\alpha}\psi \ , \qquad \overline{\psi} \to e^{-iQ\alpha}\overline{\psi} \ , \tag{18}$$

corresponding to $\Delta\psi = iQ\psi$. The corresponding conserved current is

$$j^\mu = -Q\,\overline{\psi}\gamma^\mu\psi \ . \tag{19}$$

A particularly important set of continuous transformations are the spacetime translations,

$$x^\lambda \to x^\lambda - a^\lambda \ . \tag{20}$$

For nearly all the theories we will study in this course, these translations will be symmetries of the system. For now, let's look specifically at our simple scalar theory. It is easiest to think of the translations as an active shift of the system: $\phi(x) \to \phi'(x) = \phi(x+a)$ (with the integrals and derivatives in the action unchanged, possible because we integrate over all spacetime in the action). For infinitesimal a^λ, we have

$$\phi'(x) = \phi(x) + a^\lambda \partial_\lambda \phi \ . \tag{21}$$

Applying this to the Lagrangian, we find

$$\mathscr{L}(\phi') = \mathscr{L}(\phi) + a^\lambda \partial_\mu \left(\delta^\mu_\lambda \mathscr{L}\right) \ . \tag{22}$$

Thus, spacetime translations are a symmetry of our theory with $K^\mu_\lambda = \delta^\mu_\lambda \mathscr{L}$. Applying our general result, the corresponding conserved currents are

$$j^\mu_\lambda = \partial^\mu \phi \, \partial_\lambda \phi - \delta^\mu_\lambda \mathscr{L} \ . \tag{23}$$

At this point, let us emphasize that we have just considered four different symmetries at once; each value of $\lambda = 0, 1, 2, 3$ corresponds to a different transformation.[1] In contrast, μ labels the spacetime index that always arises on the current. However, since $\eta_{\mu\nu} j^\nu_\lambda = j_{\mu\lambda} = j_{\lambda\mu}$ in this case, we can afford to be a bit careless with the indices.

For the specific case of time translations, we should take $\lambda = 0$. The corresponding charge is

$$\int d^3x \ j^0_0 = \int d^3x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right] \ . \tag{24}$$

This is just the Hamiltonian H of the system. Thus, invariance under time translations corresponds to energy conservation, $\dot{H} = 0$. Similarly, for spatial translations the related charge is

$$\int d^3x \, j^0_i = \int d^3x \ (\partial_t \phi) \, \partial_i \phi \ , \tag{25}$$

corresponding to a conserved spatial momentum P_i. Given the physical interpretation of j^μ_λ, it is given a special symbol

$$j^{\mu\nu} = T^{\mu\nu} \ , \tag{26}$$

and is called the *energy-momentum tensor*. The corresponding charges are usually combined into a single conserved 4-vector, $P^\mu = \int d^3x \, j^{0\mu} = (H, \vec{P})$.

[1] This is why we used λ instead of ν.

e.g. 7. Shifts

Any theory whose Lagrangian depends only on derivatives of a field has a symmetry under shifts of that field. For example,

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 \ , \qquad \phi \to \phi + \alpha \ . \tag{27}$$

In this case, $\Delta\phi = 1$, and the current is

$$j^\mu = \eta^{\mu\nu}\partial_\nu\phi \ . \tag{28}$$

This kind of symmetry will be important for theories in which a process called *spontaneous symmetry breaking* occurs.

2 Gauge Invariance and QED

Recall the QED Lagrangian:

$$\mathscr{L} = \sum_i \left[\, \overline{\psi}_i i\gamma^\mu(\partial_\mu + ieQ_iA_\mu)\psi_i - m\overline{\psi}_i\psi_i \,\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ . \tag{29}$$

This theory has a continuous symmetry under the transformations

$$\begin{cases} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu \end{cases} \tag{30}$$

This works provided the transformation parameter α has the same value everywhere.

Consider next what happens if we allow the transformation parameter to vary over spacetime: $\alpha = \alpha(x)$. Doing so, we find that the transformation above is no longer a symmetry of the theory. In particular,

$$\overline{\psi_i}i\gamma^\mu\partial_\mu\psi_i \to \overline{\psi}_i i\gamma^\mu\partial_\mu\psi_i + \overline{\psi}i\gamma^\mu(iQ_i\partial_\mu\alpha)\psi_i \ . \tag{31}$$

Evidently the transformation of Eq. (30) is *not* a symmetry of the theory for non-constant parameters $\alpha(x)$ due to the derivative acting on it.

The invariance of the theory under spacetime-dependent transformations of this form is restored if the vector field also transforms according to:

$$\begin{cases} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{cases} \tag{32}$$

To see, note that the combined transformations imply that

$$(\partial_\mu + ieQ_iA_\mu)\psi_i := D_\mu\psi_i \to e^{iQ_i\alpha}D_\mu\psi_i, \tag{33}$$

and therefore $\overline{\psi}_i i\gamma^\mu D_\mu \psi_i$ is invariant under the transformation for arbitrary $\alpha(x)$. The differential operator D_μ is sometimes called a *covariant derivative.* It is also not hard to check that this shift in the photon field does not alter the photon kinetic field:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{e}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha = F_{\mu\nu} + 0 \ . \tag{34}$$

Thus, QED is invariant under the combined transformations of Eq. (32) for any reasonable arbitrary function $\alpha(x)$.

At first glance this invariance might just seem like a clever trick, but the river beneath these still waters runs deep. Thinking back to regular electromagnetism (of which QED is just the quantized version), one often deals with scalar and vector potentials. These potentials are not unique and are therefore not observable (for the most part), and the true "physical" quantities are the electric and magnetic fields. The vector field A^μ in QED, corresponding to the photon, is identified with these potentials according to

$$A^\mu = (\phi, \vec{A}), \tag{35}$$

where ϕ and $\vec{A}$ are the usual scalar and vector potentials. This is justified by the equations of motion derived from the QED Lagrangian provided we also identify

$$F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk} B^k, \tag{36}$$

with the electric and magnetic fields. With this identification, the transformations of Eq. (32) coincide with the usual "gauge" transformations you should have encountered in electromagnetism. Sometimes we call A^μ the gauge boson and the operation of Eq. (32) a gauge transformation.

Keeping in mind the story from electromagnetism, the interpretation of the quantum fields in QED is that ***only those quantities that are invariant under the transformations of Eq.*** (32) ***are physically observable.*** In particular, the vector field A_μ that represents the photon is not itself an observable quantity, but the gauge-invariant field strength $F_{\mu\nu}$ is. Put another way, the field variables we are using are redundant, and the transformations of Eq. (32) represent an *equivalence relation*: any two set of fields (ψ, A_μ) related by such a transformation represent the same physical configuration. Sometimes the invariance under Eq. (32) is called a local or gauge symmetry, but it is not really a symmetry at all. A true symmetry implies that different physical configurations have the same properties. Gauge invariance is instead a statement about which configurations are physically observable.

Gauge invariance is also sensible if we consider the independent polarization states of the photon, of which there are two. The vector field A_μ represents the photon, but it clearly has four independent components. Of these, one component (corresponding to configurations of the form $A_\mu = \partial_\mu \phi$ for some scalar ϕ) is already non-dynamical on account of the form of the vector kinetic term. Invariance under gauge transformations effectively removes the additional longitudinal polarization leaving behind only the two physical transverse polarization states. Note as well that if the photon had a mass term, $\mathscr{L} \supset m^2 A_\mu A^\mu/2$, the theory would no longer be gauge invariant. Instead, the longitudinal polarization mode would enter as physical degree of freedom. Equivalently, gauge invariance forces the photon to be massless.

In the discussion above we started with the QED Lagrangian and showed that it was gauge-invariant. However, the modern view is to take gauge invariance as the fundamental principle. Indeed, the only way we know of to write a consistent, renormalizable theory of interacting vector fields is to have an underlying gauge symmetry. For QED, we could have started with a local gauge invariance for a charged fermion field and built up the rest of the Lagrangian based on this requirement. In this context, the vector field is needed to allow us to define a sensible derivative operator on the fermion field, which involves taking a difference of two fields at different spacetime points with apparently different transformation properties, and corresponds to something called a *connection*.[2] Gauge invariance completely fixes the photon-fermion interactions, illustrating why it is so powerful. We will see shortly that gauge invariance is even more powerful when the underlying symmetry transformations are more complicated.

3 Symmetries and Groups

To go beyond QED, it will be useful to have a more formal mathematical description of symmetry transformations. In particular, symmetry transformations obey the mathematical properties of a *group*, and it is worth spending a bit of time discussing them.[3] Along the way, we will be led to introduce the concepts of *representations* and *Lie groups*.

A group G is a set of objects together with a multiplication rule such that:

1. if $f,\, g \in G$ then $h = f\cdot g \in G$ (closure)
2. $f\cdot(g\cdot h) = (f\cdot g)\cdot h$ (associativity)
3. there exists an identity element $1 \in G$ with $1\cdot f = f\cdot 1 = f$ for any $f \in G$ (identity)
4. for every $f \in G$ there exists an inverse element f^{-1} such that $f\cdot f^{-1} = f^{-1}\cdot f = 1$ (invertibility)

A group can be defined via a multiplication table that specifies the value of $f\cdot g$ for every pair of elements $f,\, g \in G$. An *Abelian* group is one for which $f\cdot g = g\cdot f$ for every pair of $f,\, g \in G$. A familiar example of an Abelian group is the set of rotations in two dimensions. In contrast, the set of rotations in three dimensions is non-Abelian.

For the most part, we will be interested in symmetry transformations that act linearly on quantum fields,

$$\phi_i \to \phi_i' = U_{ij}\phi_j \ , \tag{37}$$

[2] See Ref. [2] for a nice explanation of these slightly cryptic comments.

[3] Much of this discussion is based on Refs. [3, 2], both of which provide a much more detailed account of the topics covered here.

where U_{ij} is independent of the fields. As a result, we will usually work with matrix *representations* of groups. Groups themselves are abstract mathematical objects. A representation of a group is a set of $n \times n$ matrices $U(g)$, one for each group element, such that:

1. $U(f)U(g) = U(f \cdot g)$
2. $U(1) = \mathbb{I}$, the identity matrix.

Note that these conditions imply that $U(f^{-1}) = U^{-1}(f)$. The value of n is called the dimension of the representation. For any group, there is always the *trivial* representation where $U(g) = \mathbb{I}$ for every $f \in G$. Note that a representation does not have to faithfully reproduce the full multiplication table. A representation is said to be *unitary* if all the representation matrices can be taken to be unitary ($U^\dagger = U^{-1}$).

e.g. 8. Rotations in two dimensions

This group is formally called $SO(2)$ and can be defined as an abstract mathematical object. Any group element can be associated with a rotation angle θ. The most familiar representation is in terms of 2×2 matrices,

$$D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{38}$$

Of course, there is also the trivial representation, $D(\theta) = 1$.

Our focus will be primarily on continuous transformations. These correspond to what are called *Lie groups*, which are simply groups whose elements can be parametrized in terms of a set of continuous variables $\{\alpha^a\}$, where a labels the set of coordinates needed. We can (and will) always choose these coordinates (near the identity) such that the point $\alpha^a = 0$ corresponds to the identity element of the group. Thus, for any representation of the group, we have for infinitesimal transformations near the identity

$$U(\alpha^a) = \mathbb{I} + i\alpha^a t^a + \mathcal{O}(\alpha^2). \tag{39}$$

The matrices t^a are called *generators* of the representation. Finite transformations can be built up from infinitesimal ones according to

$$U(\alpha^a) = \lim_{p\to\infty} (1 + i\alpha^a t^a/p)^p = e^{i\alpha^a t^a}. \tag{40}$$

This is nice because it implies that we only need to sort out a finite set of generators when discussing the representation of a Lie group rather than the infinite number of group elements.

A set of generator matrices $\{t^a\}$ can represent a Lie group provided they satisfy a *Lie algebra*. Besides being able to add and multiply them, they must also satisfy the following conditions:

1. $[t^a, t^b] = if^{abc}t^c$ for some constants f^{abc}

2. $[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0$ (Jacobi Identity)

The first condition is needed for the closure of the group (*i.e.* $\exp(i\alpha^a t^a)\exp(i\beta^a t^a) = \exp(i\lambda^a t^a)$ for some λ^a) while the second is required for associativity. In fact, we can define a Lie group abstractly by specifying the *structure constants* f^{abc}. Most of the representations we will work with are unitary, in which case the structure constants are all real and the generators t^a are Hermitian.

The great thing about working with linear generators t^a is that we can choose a nice basis for them. This is equivalent to choosing a nice set of coordinates for the Lie group. In particular, it is always possible to choose the generators t^a_r of any representation r such that

$$tr(t^a_r t^b_r) = T_2(r)\delta^{ab}. \tag{41}$$

The constant $T_2(r)$ is called the Dynkin index of the representation. We will always implictly work in bases satisfying Eq. (41), and we will concentrate on the case where the index is strictly positive. If so, the corresponding Lie group is said to be *compact* and is guaranteed to have finite-dimensional unitary representations. (A familiar non-compact example is the Minkowski group.)

There are only a few classes of compact Lie groups. The *classical* groups are:

- $U(1)$ = phase transformations, $U = e^{i\alpha}$
- $SU(N)$ = set of $N \times N$ unitary matrices with $det(U) = 1$
- $SO(N)$ = set of orthogonal $N \times N$ matrices with $det(U) = 1$
- $Sp(2N)$ = set of $2N \times 2N$ matrices that preserve a slightly funny inner product.

In addition to these, there are the *exceptional* Lie groups: E_6, E_7, E_8, F_4, G_2. In studying the Standard Model, we will focus primarily on $U(1)$ and $SU(N)$ groups.

e.g. 9. $SU(2)$

This is the prototypical Lie group, and should already be familiar from what you know about spin in quantum mechanics. By definition, the corresponding Lie algebra has three basis elements which satisfy

$$[t^a, t^b] = i\epsilon^{abc}t^c \tag{42}$$

The basic *fundamental* representation of $SU(2)$ is in terms of Pauli matrices: $t^a = \sigma^a/2$. Since $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, it's clear that this is a valid representation of the algebra. You might also recall that any $SU(2)$ matrix can be written in the form $U = \exp(i\alpha^a\sigma^a/2)$.

Some useful and fun facts about compact Lie algebras:

- Except for $U(1)$, we have $tr(t^a) = 0$ for all the classical and exceptional Lie groups.

- Number of generators = $d(G)$

$$d(G) = \begin{cases} N^2 - 1; & SU(N) \\ N(N-1)/2; & SO(N) \\ 2N(2N+1)/2; & Sp(2N) \end{cases} \tag{43}$$

- A representation (= rep) is *irreducible* if it cannot be decomposed into a set of smaller reps. This is true if and only if it is impossible to simultaneously block-diagonalize all the generators of the rep. Irreducible representation = irrep.

- If one of the generators commutes with all the others, it generates a $U(1)$ subgroup called an Abelian factor: $G = G' \times U(1)$.

- If the algebra cannot be split into sets of mutually commuting generators it is said to be *simple*. For example, $SU(5)$ is simple (as are all the classical and exceptional Lie groups given above) while $SU(3) \times SU(2) \times U(1)$ is not simple. In the latter case, all the $SU(3)$ generators commute with all the $SU(2)$ generators and so on.

- A group is *semi-simple* if it does not have any Abelian factors.

- With the basis choice yielding Eq. (41), one can show that the structure constants are completely anti-symmetric.

- The *fundamental* representation of $SU(N)$ is the set of $N \times N$ special unitary matrices acting on a complex vector space. This is often called the **N** representation. Similarly, the fundamental representation of $SO(N)$ is the set of $N \times N$ special orthogonal matrices acting on a real vector space.

- The *adjoint* (= A) representation can be defined in terms of the structure constants according to

$$(t_A^a)_{bc} = -if^{abc} \tag{44}$$

  Note that on the left side, a labels the adjoint generator while b and c label its matrix indices.

- Given any rep t_r^a, the conjugate matrices $-(t_r^a)^*$ give another representation, unsurprisingly called the conjugate representation. A rep is said to be real if it is unitarily equivalent to its conjugate. The adjoint rep is always real.

- The Casimir operator of a rep is defined by $T_r^2 = t_r^a t_r^a$ (with an implicit sum on a). One can show that T_r^2 commutes with all the t_r^a. For an irrep (=irreducible representation) of a simple group, this implies that

$$T_r^2 = C_2(r)\mathbb{I}, \tag{45}$$

  for some positive constant $C_2(r)$.

- It is conventional to fix the normalization of the fundamental of $SU(N)$ such that $T_2(\mathbf{N}) = 1/2$. Once this is done, it fixes the normalization of all the other irreps. In particular, it implies that for $SU(N)$, $C_2(\mathbf{N}) = (N^2-1)/2N$, $T_2(A) = N = C_2(A)$.

4 Aside: Representations of Lorentz and Spin

A really important symmetry group of relativistic QFTs is Poincaré, the combined group of spacetime translations and Lorentz transformations. The quantum mechanical property of spin of elementary particles can be understood as a consequence of this invariance. We will only give a rough sketch of how this works here, but more details can be found in Refs. [6, 7].

The Poincaré group is a Lie group with ten generators: four spacetime translations, three rotations, and three Lorentz boosts. This symmetry is usually incorporated into QFTs by using fields that transform under definite representations of Poincaré. In contrast to the Lie groups discussed above, the Poincaré group is not compact, and its representations are typically not unitary. The spacetime translation part of the group is handled by using fields that are functions of spacetime, which allow them to provide what is called a *coordinate representation* of the subgroup. For the Lorentz portion of the group, the three simplest representations are the scalar ($s = 0$), Weyl fermion ($s = 1/2'$), and vector ($s = 1'$) we have already seen.

In addition to using field operators that transform as definite representations of Poincaré, we also want to use state vectors in the quantum Hilbert space that are representations of the group. This is a bit more complicated for states because the representations in this case have to be unitary. For massive states, $P^2 = M^2 > 0$, these representations are labelled unambiguously by their total momentum and spin, with the spin part corresponding to the familiar $SU(2)$ group in the particle rest frame. For massless states, $P^2 = 0$, the spin part corresponds instead to the group $ISO(2)$, and these states are characterized by their total momentum and helicity.

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