P528 Notes #1: Review of QFT

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Quantum Field Theory (QFT) is the best tool we know of to describe the physics of elementary particles. It is the basic language used in elementary particle physics, superstring theory, and many branches of condensed matter physics. Learning QFT has the reputation of being difficult, but it is more accurate to say that it is time-consuming. Since I can't cram a whole year's worth of material into a single class, I will try to give you a general flavour of the topic. For a list of useful textbooks, take a look at Refs. [1, 2, 3, 4, 5, 6, 7]. I also strongly encourage you to take a full QFT course at some point in your graduate career.

1 General Overview

QFT is nothing more than ordinary quantum mechanics formulated in a relativistically invariant way and applied to continuous *field* systems. The only difference from the more-familiar one-particle quantum mechanics is what we identify as the underlying degrees of freedom, which in this case are the fields. It might seem strange to use continuous fields to describe discrete objects like particles. When a field is quantized, however, there often appear discrete quantum excitations that can be interpreted as particles. This interpretation is justified *a posteriori* by its excellent agreement with experiment. On the other hand, ordinary one-particle quantum mechanics doesn't get along well with (special) relativity because it does not account for particle creation and annihilation at high energies. Quantum field theory avoids this problem, and reduces to one-particle quantum mechanics in the appropriate limit. For a nice alternative point of view on why we use QFT to describe elementary particles, read Ch.1 of Burgess & Moore [8].

We usually define a QFT in terms of an *action* that depends on a set of fields $\{\phi_i\}$. For the cases of interest to us, the action can be written in the form

$$S[\phi_i] = \int d^4x \, \mathscr{L}(\phi_i(x)). \tag{1}$$

The function $\mathscr{L}(\phi_i)$ is the Lagrange density, but we will usually just call it the *Lagrangian*. Some comments about the action [2]:

- We will usually assume that $\phi_i \to 0$ and $\partial_\mu \phi_i \to 0$ as $\max\{|t|, |x|, |y|, |z|\} \to \infty$ so that we can ignore total derivatives in the action: $\int d^4x \, \partial_\mu F(\phi_i) = 0$ for any polynomial function $F(\phi_i)$. The vanishing of fields at the spacetime boundary is usually a necessary condition for a given field configuration to have finite energy.
- S depends on x^μ only through the fields $\phi_i(x)$. This implies invariance under spacetime translations.

- S should also be invariant under Lorentz boosts and rotations. These take the form

$$x^\mu \rightarrow x'^\mu(x) = \Lambda^\mu{}_\nu x^\nu \tag{2}$$
$$\phi_A(x) \rightarrow \phi'_A(x') = [M(\Lambda)]_A{}^B \phi_B(\Lambda^{-1}x') \ , \tag{3}$$

  where Λ defines a Lorentz transformation and $M(\Lambda)$ is a transformation matrix that depends on Λ. As we will discuss later on, the matrices $M(\Lambda)$ form a *representation* of the Lorentz group. Applying this to the action and using $d^4x' = d^4x$, Lorentz invariance requires that $\mathscr{L}(\phi) \rightarrow \mathscr{L}'(\phi') = \mathscr{L}(\phi')$ up to total derivatives. In other words, the Lagrangian after the Lorentz transformation must take the same functional form as the original Lagrangian.

- The action is *local*, in that it depends only on functions (and derivatives) of fields at the same spacetime point. (*e.g.* There are no terms like $\int d^4x \int d^4y\, \phi(x)\phi(y)$ in the action.) This is needed if we want our theory to be causal.

- Dimensions: $[S] = 0$, $[d^4x] = -4$, so we need $[\mathscr{L}] = +4$.

- S needs to be real for the theory to be unitary.

An action that satisfies these conditions can potentially give rise to a well-defined, Lorentz-invariant QFT. Because of Lorentz invariance, we can work with fields that have well-defined transformation properties under Lorentz transformations. For sufficiently weak interactions, this allows us to identify fields with particles of definite spins (or helicities):

Scalar $(\mathbf{s = 0})$**:** $\phi'(x') = \phi(x)$
Fermion $(\mathbf{s = 1/2'})$**:** $\psi'_a(x') = M_a{}^b \psi_b(x)$ (The indices here are *spinor* indices.)
Vector $(\mathbf{s = 1'})$**:** $A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)$

A handy rule of thumb is that the Lagrangian will be Lorentz invariant if all the Lorentz indices on the fields are properly contracted.

In practice, we want to use QFT to compute things that can be compared to experiment. The standard technique for this is to make a perturbative expansion of the QFT around the non-interacting free-field theory consisting only of quadratic (and lower) powers of the fields. This involves three steps:

1. Start with the quadratic (and lower) terms in the Lagrangian and extract from them the kinetic and mass terms.

2. For this, redefine the field variables to put the kinetic terms in *canonical* form and diagonalize the mass matrices.

3. Add the terms higher than quadratic (in terms of the redefined and now-canonical/diagonal fields) and identify the interactions they correspond to.

We will illustrate the first two steps for scalar, fermion, and vector fields below. With this in place, we will add interactions and discuss how to compute things using Feynman rules.

2 Scalar Fields

The simplest kind of field is a real Lorentz scalar, for which the transformation matrix of Eq. (3) is trivial (*i.e.* the unit matrix):

$$\phi(x) \to \phi'(x') = \phi(x) \ . \tag{4}$$

This implies that they describe particles of spin $s = 0$. The basic quadratic Lagrangian for a real scalar field is [2]

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \ , \tag{5}$$

where $(\partial\phi)^2 = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The first piece here is the kinetic term and the second is the mass term. The kinetic term has a canonical normalization, corresponding to a momentum-space propagator of $i/(p^2 - m^2 + i\epsilon)$. According to the rules above, this Lagrangian describes a single particle species of physical mass $m_{phys} := \sqrt{m^2}$. Adding cubic or higher-order terms to the Lagrangian would introduce self-interactions between the ϕ particles.

e.g. 1. Non-canonical kinetic term
Suppose we have the Lagrangian

$$\mathscr{L} = \frac{1}{2}Z(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \ , \tag{6}$$

for some positive constant Z. For $Z \neq 1$, this deviates from the canonical normalization for a real scalar because it leads to a propagator of $iZ^{-1}/(p^2 - m^2/Z + i\epsilon)$. This can be repaired by using different field variables, $\phi(x) = Z^{-1/2}\widetilde{\phi}(x)$, in terms of which the Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}(\partial\widetilde{\phi})^2 - \frac{1}{2}(m^2/Z)\widetilde{\phi}^2 \ . \tag{7}$$

The kinetic term in terms of the new variable $\widetilde{\phi}$ is now canonical. Following our rules, the physical mass of the corresponding particle is $m_{phys} = \sqrt{m^2/Z}$. Note that this only makes sense if $Z > 0$. For $Z < 0$, the theory is inconsistent.

e.g. 2. Multiple scalars and mass mixing
Suppose we now have n real scalars ϕ_i $(i = 1, 2, \ldots, n)$ with

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\delta_{ij}\eta^{\mu\nu}(\partial_\mu\phi_i\,\partial_\nu\phi_j) - \frac{1}{2}(M^2)_{ij}\phi_i\phi_j \\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^t\partial_\nu\phi - \frac{1}{2}\phi^t M^2\phi \ , \end{aligned} \tag{8}$$

where M^2_{ij} is a real symmetric matrix and a sum over repeated indices (Lorentz and i, j) is implied. In the second line, we have simply rewritten the first line in terms of a matrix notation with $\phi^t = (\phi_1, \ldots, \phi_n)$ is an n-component row vector. The kinetic terms of all the

scalars are canonical, but in general the $n \times n$ matrix M^2 is not diagonal. Fortunately, any real symmetric matrix can be diagonalized by an orthogonal transformation, $\mathcal{O}^t M^2 \mathcal{O} = diag(m_1^2, \ldots, m_n^2)$ for some eigenvalues m_i^2. We can use this fact to make sense of the theory by transforming to nicer field variables $\widetilde{\phi}$. Defining $\phi = \mathcal{O}\widetilde{\phi}$ and plugging into the original Lagrangian, we find

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\widetilde{\phi}^t\partial_\nu\widetilde{\phi} - \frac{1}{2}\widetilde{\phi}^t diag(m_1^2, \ldots, m_n^2)\widetilde{\phi} \qquad (9) \\ &= \sum_i \left[\frac{1}{2}(\partial\widetilde{\phi}_i)^2 - \frac{1}{2}m_i^2(\widetilde{\phi}_i)^2\right] . \end{aligned}$$

Thus, we obtain a theory with n independent scalar particles with physical masses $m_{i,phys} = \sqrt{m_i^2}$. Note that this only makes sense if all the eigenvalues m_i^2 are non-negative; we will learn how to deal with negative eigenvalues later on. More generally, with n real scalars the kinetic term can also have a mixing matrix Z_{ij} (instead of the δ_{ij} we had above). To deal with this, first diagonalize the kinetic term, then rescale each of the fields to get canonical kinetic terms for all fields, and finally diagonalize the resulting mass matrix.

e.g. 3. Complex scalar field
Consider a theory with two real scalars ϕ_1 and ϕ_2 with the same mass. In this case, we can rewrite the basic quadratic Lagrangian in terms of a single complex scalar $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$:

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right) \qquad (10) \\ &= |\partial\Phi|^2 - m^2|\Phi|^2 , \qquad (11) \end{aligned}$$

where $|\partial\Phi|^2 = \eta^{\mu\nu}\partial_\mu\Phi^*\,\partial_\nu\Phi$. This form, without the factor of 1/2, is the canonical kinetic form for a complex scalar with physics mass $m_{phys} = \sqrt{m^2}$. The choice of representing the degrees of freedom in terms of two mass-degenerate real fields or one complex field is a matter of convenience. However, when the theory has a symmetry under rephasing, $\Phi \to e^{-i\alpha}\Phi$, the complex form is usually nicer.

3 Fermion Fields

Particles with odd-integer spins are fermions. The most familiar example in four spacetime dimensions is the four-component Dirac fermion, transforming under Lorentz according to

$$\psi_a(x) \to \psi'_a(x') = [M(\Lambda)]_{ab}\psi_b(x) , \qquad (12)$$

where a, $b = 1, 2, 3, 4$ are called *Dirac indices*, $M(\Lambda)$ is a 4×4 matrix, and summation over the repeated Dirac index b is implied. These four components of ψ are related to the two spin states of a spin $s = 1/2$ particle plus the two spin states of its distinct antiparticle.

Before we write the basic Lagrangian for a Dirac fermion, let us introduce some notation [2]. First, generalize to the 2×2 Pauli matrices to

$$\sigma^0 = \mathbb{I}, \quad \sigma^i = \sigma^{1,2,3} \tag{13}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{14}$$

Recall that

$$\sigma^i \sigma^j = \delta^{ij}\mathbb{I} + i\epsilon^{ijk}\sigma^k . \tag{15}$$

Let us also define

$$\sigma^\mu = (\mathbb{I}, \vec{\sigma}) , \quad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma}) . \tag{16}$$

In terms of these, we define the 4×4 Dirac matrices in the so-called *chiral representation* by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{17}$$

These satisfy the familiar relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{18}$$

We will also define

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\lambda\kappa}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma\kappa . \tag{19}$$

In the chiral representation, one finds

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} . \tag{20}$$

We will also encounter the chiral projectors $P_L = (1 - \gamma^5)/2$ and $P_R = (1 + \gamma^5/2)$.

The basic Lagrangian for a Dirac fermion is

$$\mathscr{L} = \overline{\psi} i\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi , \tag{21}$$

where the first term is the kinetic piece in canonical normalization and the second is a mass term with $m_{phys} = |m|$. Note that this is written in matrix notation with $\overline{\psi} = \psi^\dagger\gamma^0$. Using the properties of the Lorentz transformation matrices $[M(\Lambda)]_{ab}$, it can be shown that this Lagrangian is Lorentz invariant.

It is instructive to rewrite the basic Lagrangian of Eq. (21) in a couple of different ways. First, we can label the Dirac component structure explicitly, which gives

$$\mathscr{L} = \psi^*_a(\gamma^0)_{ab}\gamma^\mu_{bc}\partial_\mu\psi_c - m\,\psi^*_a(\gamma^0)_{ac}\psi_c , \tag{22}$$

where repeated indices are summed over. This form is messy, and is not usually written explicitly, but it is good to know how the Dirac indices connect up.

A second useful rewriting of Eq. (21) is in terms of the chiral components of ψ, inspired by the 2×2 block structure of the gamma matrices. Let us define

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{23}$$

In a mild abuse of notation, we will also write

$$\psi_L := P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} , \qquad \psi_R := P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} . \tag{24}$$

In terms of the *chiral components* ψ_L and ψ_R, the basic Lagrangian of Eq. (21) becomes

$$\mathscr{L} = \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R - m \left(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L \right) . \tag{25}$$

Note that the kinetic term does not mix the chiral components but the mass term does. In the absence of the mass term, it is possible to write a consistent theory with only one or the other of the chiral components. The labels L and R correspond to the fact that in the massless limit, the ψ_L field describes a fermion of left-handed helicity plus an antiparticle of right-handed helicity, where *helicity* refers to the angular momentum orientation relative to the direction of motion. Similarly, ψ_R describes a particle with right-handed helicity plus an antiparticle of left-handed helicity.

4 Vector Fields

A vector field is one that transforms like a Lorentz 4-vector:

$$A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) . \tag{26}$$

Even though $A_\mu(x)$ has four independent components, this field does not describe four independent particles. A massive vector corresponds to a spin $s = 1$ particle species with three components. For a massless vector, there are only two independent *helicities*. Either way, the vector field has more components than physical states, so not all the components are physically independent [9, 10].

The basic Lagrangian for a vector field is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu , \tag{27}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called the *field strength*. The first piece is the canonically normalized kinetic term and the second is a mass term corresponding to a physical mass $m_{phys} = \sqrt{m^2}$. The kinetic term here might look a bit funny compared to what we had for

scalars, but it becomes more appealing after integrating by parts (and dropping the surface term in the action):

$$\mathscr{L} = \frac{1}{2} A^\mu (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + m^2\eta_{\mu\nu}) A^\nu \ . \tag{28}$$

Except for the $\partial_\mu\partial_\nu$ term and the Lorentz indices, this looks just like what one would obtain for the scalar field Lagrangian of Eq. (5) after integrating by parts.

In the massive case, $m \neq 0$, the four component object A^μ contains spin $s = 0$ and $s = 1$ pieces. The $s = 0$ part corresponds to configurations of the field that can be written as a derivative of a scalar, $A_\mu = \partial_\mu \phi$ for some ϕ. To keep only the $s = 1$ part, we need an additional constraint, which turns out to be

$$\partial_\mu A^\mu = 0 \ . \tag{29}$$

Going back to Eq. (28), the derivative pieces already conspire to annihilate a potential scalar component of A^μ:

$$0 = (\eta^{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\, \partial^\nu \phi \ . \tag{30}$$

For the mass term, we find after integrating by parts

$$m^2\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \to -m^2\phi(\partial^2\phi) \ . \tag{31}$$

This vanishes for $0 = \partial_\mu A^\mu \to \partial_\mu\partial^\mu\phi$.

The situation for the massless case is a bit more subtle. Now, the theory (with the funny kinetic term) has an invariance under

$$A_\mu \to A_\mu + \partial_\mu \alpha \ , \tag{32}$$

for any smooth $\alpha(x)$. The interpretation now is that any two field configurations related by such a transformation are physically equivalent. This effectively removes two degrees of freedom from the vector allowing it to describe two physical helicity states. Note that this invariance should remind you of classical electromagnetism, where $A^\mu = (\phi, \vec{A})$ is identified with the scalar and vector potentials. Recall that these objects do not correspond to unique physical configurations. Instead, it is the electric and magnetic fields derived from them that are unambiguous and physically relevant. We will discuss this *gauge invariance* in more detail later in the course.

5 Interactions and Feynman Rules

Thus far we have concentrated on the quadratic parts of QFT Lagrangians, and we showed how they can be used to identify the physical particles in the theory. Higher-order terms in the Lagrangian correspond to interactions among these physical excitations. Such interactions lead to particle scattering and decay, which correspond to the main physical observables

relevant for the study of subatomic particles. A powerful tool for computing these observables are *Feynman rules*, and we discuss how to derive and apply them here.

A given QFT is defined by its action (and a few more things we do not need to worry about for now). The Feynman rules for a theory can be derived from its Lagrangian. There are two main steps. First, following the procedures derived above, one identifies the physical excitations of the theory (*e.g.* particles) in the absence of interactions from the quadratic terms in the Lagrangian. These fix the *propagators* for internal legs in a Feynman diagram as well as the *polarization vectors* for the external legs of the diagram. And second, the Lagrangian terms above quadratic order are identified with interactions, and *vertex factors* are derived for each of them. Such vertices connect the internal and external legs in Feynman diagrams. Feynman diagrams correspond to an expansion in powers of these vertices.

With the Feynman rules in hand, they can be used to compute the *transition matrix element* (also called the *scattering amplitude*) $\mathcal{M}$ for a given process. Recall from `notes-00` that such matrix elements are squared and integrated over to obtain cross sections and decay rates. Note that a given matrix element depends on the particle types, momenta, and spins of all the particles in the initial and final states. Once these are specified, computing the corrsponding amplitude involves the following steps:

1. Draw all possible Feynman diagrams up to the desired level of expansion in powers of the interaction vertices. (We will mostly stick to the lowest non-trivial order here.) To do so, fix the external legs and connect them up in all possible ways using the vertices of the theory and possibly some internal legs.

2. For each diagram, use the Feynman rules to find the mathematical expression for each diagram. For closed loops, integrate over internal loop momentum and add a factor of (-1) if it is a fermion loop.

3. Sum the expressions for each of the diagrams. Include any necessary symmetry factors for identical initial or final states. In the case of fermions, diagrams that differ only by the interchange of two identical fermion lines come have a relative factor of (-1).

4. The result of all this is $(-i)$ times the matrix element.

All this is abstract, so it will help to illustrate these procedures with some specific examples.

e.g. 4. $\lambda\,\phi^4$ Theory

Consider the theory of a real scalar field ϕ with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{33}$$

For $\lambda = 0$ this corresponds to a real scalar ($s = 0$) field with mass m. The quartic term defines a self-interaction among ϕ particles and the dimensionless coupling λ determines its strength. The corresponding Feynman rules are shown in Fig. 1.[1]

[1] Note that we are being sloppy and omitting the $\imath\epsilon$ factors in the propagators here.

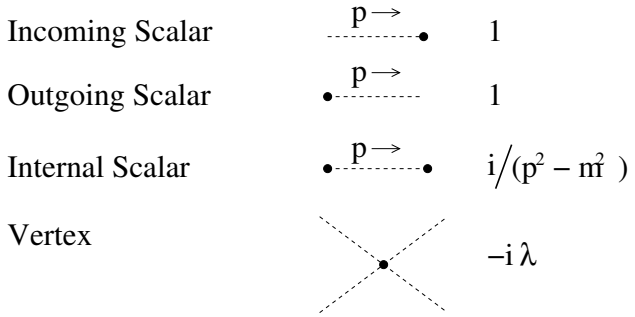

Figure 1: Feynman rules for the $\lambda\phi^4$ theory.

The most simple process in this theory is $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$ scattering. In Fig. 2 we show the contributions to matrix element at linear and quadratic order in λ. The corresponding contributions to the matrix element are

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \ldots \tag{34}$$

with the first term corresponding to the order λ^1 diagram

$$-i\mathcal{M}_1 = -i\lambda \tag{35}$$

and the second term corresponds to to the order λ^2 diagrams,

$$\begin{aligned} -i\mathcal{M}_2 \;\; = \;\; & \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \, \frac{1}{q^2 - m^2} \Big[\frac{1}{(p_1 + p_2 + q)^2 - m^2} \\ & + \frac{1}{(p_1 - p_3 + q)^2 - m^2} + \frac{1}{(p_1 - p_4 + q)^2 - m^2} \Big] \end{aligned} \tag{36}$$

Note that there are infinitely many diagrams and that we have arranged the Feynman expansion in powers of λ. For $\lambda \ll 1$, we expect $\mathcal{M} \simeq \mathcal{M}_1$ so that the *leading order* λ^1 estimate of the matrix should be a good one. When this is not satisfied, $\lambda \gtrsim 4\pi$, the expansion in powers of λ, and the Feynman diagram approach itself, is no longer useful. In fact, when this occurs the basic excitations of the theory may no longer match up in an obvious way with those of the free theory ($\lambda \to 0$). We will see an example of this when we get to the strong force later in the course.

e.g. 5. Quantum Electrodynamics (QED)

This is the theory of charged fermions interacting with the photon. It consists of a massless vector for the photon and a set of charged fermions. The Lagrangian is [2]

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{\psi}_i [i\gamma^\mu (\partial_\mu + ieQ_i A_\mu) - m_i] \psi_i \tag{37}$$

Sometimes we will write $D_\mu = (\partial_\mu + ieQ_i A_\mu)$, which is called a *covariant derivative*. The sum on i here refers to a sum over all charged particles in the theory. For now, we will

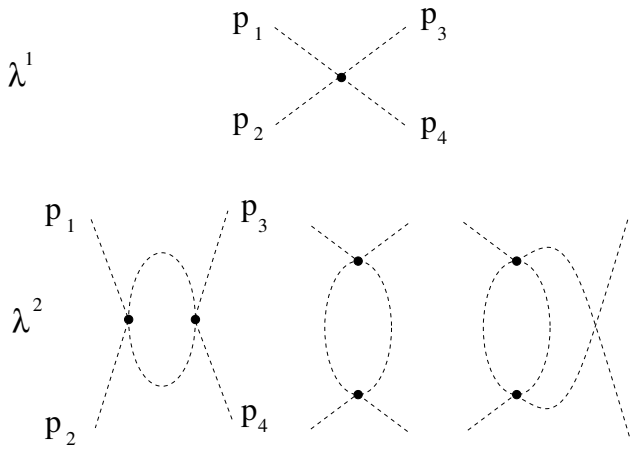

Figure 2: Feynman rules for the $\lambda\phi^4$ theory.

assume that there are only electrons ($m_e \simeq 0.511$ MeV) and muons ($m_\mu \simeq 105.7\,\mathrm{MeV}$). The only interactions in the theory are between the fermions and the photon with strength determined by the dimensionless quantity eQ. For electrons and muons we have $Q = -1$ as well as the low energy coupling value

$$\alpha \equiv \frac{e^2}{4\pi} \;\simeq\; 1/137\ . \tag{38}$$

Fortunately, this is small and thus we expect an expansion in Feynman diagrams to be useful for QED.

From this Lagrangian one can derive the Feynman rules given in Fig. 3. The quantities $u(p,s)$ and $v(p,s)$ here are 4×1 fermion and anti-fermion spin vectors for 4-momentum p and spin state s, with $\bar{u} = u^\dagger\gamma^0$. We have also written $\not{p} = p_\mu\gamma^\mu$. For photons, $\epsilon_\mu(p,\lambda)$ is the polarization 4-vector for the polarization state λ. Recall that photons have two independent transverse polarizations.

A simple process in this theory is $e^-(p_1,s_1) + e^+(p_2,s_2) \to \mu^-(p_3,s_3) + \mu^+(p_4,s_4)$. The single Feynman diagram for the process at leading order in α is shown in Fig. 4. Following the Feynman rules for QED, the amplitude is

$$-i\mathcal{M} = ie^2Q_eQ_\mu\frac{1}{p^2}(\bar{u}_3\gamma^\mu v_4)(\bar{v}_2\gamma^\nu u_1)\eta_{\mu\nu}. \tag{39}$$

Here, $p = (p_1+p_2) = (p_3+p_4)$, the subscripts label the momenta of the spinors, and the spinor indices are contracted implicitly.

Deriving Feynman rules for other theories follows the same general prescription. In particular, the propagators for internal legs and polarization vectors for external legs are universal for a given particle type. The only new thing to do (after the theory has been put into canonical form and the particle types in the free limit have been identified) is determine the interaction vertices. As long as the interaction terms in the Lagrangian do

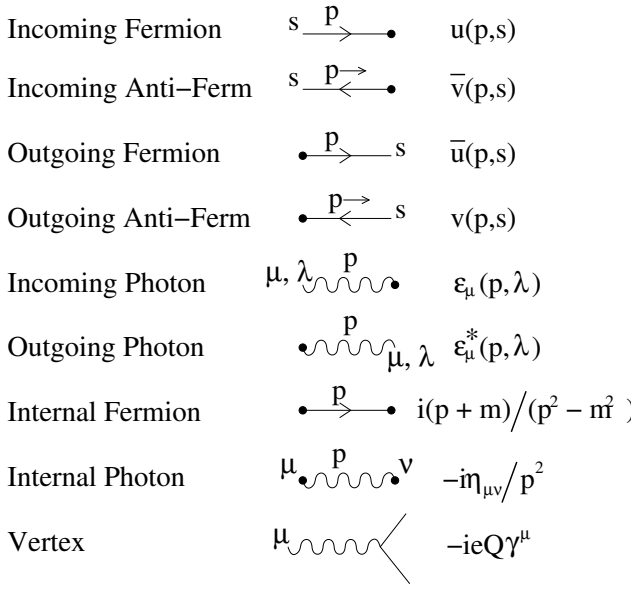

Figure 3: Feynman rules for QED.

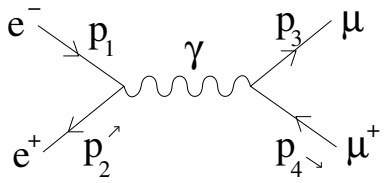

Figure 4: Leading Feynman diagram for $e^+e^- \to \mu^+\mu^-$.

not contain any derivatives there is a simple prescription to do so. The vertex with n_ϕ ingoing or outgoing ϕ scalars, n_ψ ingoing ψ fermions (or outgoing antifermions), $n_{\bar\psi}$ outgoing ψ fermions (or ingoing antifermions), and n_A ingoing or outgoing A^μ vectors is schematically

$$\text{Vertex} = i \times \frac{\partial^{(n_\phi+n_\psi+n_{\bar\psi}+n_A)}}{(\partial\phi)^{n_\phi}(\partial\psi)^{n_\psi}(\partial\bar\psi)^{n_{\bar\psi}}(\partial A_\mu)^{n_A}}\,\mathscr{L}\,\Big|_{\phi=\psi=\bar\psi=A=0} \tag{40}$$

This looks messy, but it is simple to do in practice. It can be derived from the path integral formulation of QFT [3, 6].

For the $\lambda\phi^4$ theory, the only non-vanishing vertex has four ϕ fields. Applying this prescription,

$$i \times \frac{\partial^4}{(\partial\phi)^4}\left(-\frac{\lambda}{4!}\phi^4\right)\Big|_{\phi=0} = -i\lambda\ , \tag{41}$$

which matches the result above. Note that all other possible derivatives vanish: taking more than four derivatives obviously gives zero while taking fewer than four leaves leftover powers of ϕ that vanish when it is set to zero.

Going next to QED, the only vertex corresponds to

$$i \times \frac{\partial^3}{(\partial A_\mu)(\partial\psi)(\partial\bar{\psi})} \mathscr{L}\Big|_{\psi=\bar{\psi}=A=0} = -ieQ\,\gamma^\mu \ . \tag{42}$$

Again, this matches what we had above. Note also that the vector index of A^μ in the derivative matches the value in the vertex. This follows from

$$\frac{\partial}{\partial A^\mu} A^\nu = \delta^\nu_\mu \ , \tag{43}$$

since the different components of A^μ are to be treated as independent fields. There is also something similar going on for the fermions, which carry Dirac indices, but I have hidden them for now.

References

[1] For an informal and very intuitive introduction to QFT, take a look at:
A. Zee, "Quantum field theory in a nutshell," *Princeton, UK: Princeton Univ. Pr. (2003) 518 p*

[2] This is the standard modern QFT textbook. It is probably the most comprehensive single text on the subject with excellent exercises. If you want to learn QFT and have to choose a single book, this is it. However, because it is so comprehensive, some specific topics are treated in more detail elsewhere.
M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[3] A good complement to Peskin and Schroeder:
L. H. Ryder, "Quantum Field Theory," *Cambridge, Uk: Univ. Pr. (1985) 443p*

[4] A very nice newer QFT textbook is:
M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*

[5] An even newer and also very nice QFT textbook:
M. D. Schwartz, "Quantum Field Theory and the Standard Model," *Cambridge, UK: Univ. Pr. (2013) 863 p,* http://www.schwartzqft.com/

[6] This book gives a non-standard but very clever introduction to QFT, and some basic string theory as well. There are a number of topics in this book that can be useful but are rarely covered in other texts, such as the Schrödinger representation of QFT in terms of wave functionals:
B. Hatfield, "Quantum field theory of point particles and strings," *Redwood City, USA: Addison-Wesley (1992) 734 p. (Frontiers in physics, 75)*

[7] A link to my notes from UBC PHYS-526:
D. E. Morrissey, "PHYS-526-2013",
https://particletheory.triumf.ca/dmorrissey/Teaching/PHYS526-2013/

[8] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[9] D. Tong, "Lectures on Quantum Field Theory" http://www.damtp.cam.ac.uk/user/dt281/qft.html.

[10] W. Greiner and J. Reinhardt, "Field quantization," *Berlin, Germany: Springer (1996) 440 p.*