# PHYS 528 Homework #8

Due: Mar.30, 2021

1. (Free points!) Work on your final project.

2. A toy model of regularized and renormalized integrals.

   a) Evaluate (by hand!)

   $$I_2(m^2) = \int_0^\Lambda dx\, x^3 \frac{1}{x^2+m^2} \ .$$

   *Hint:* $x^3 = x(x^2+m^2-m^2)$, *and work out the integral from there.*

   b) Compute (by hand!)

   $$I_4(m^2) = \int_0^\Lambda dx\, x^3 \left(\frac{1}{x^2+m^2}\right)^2 \ .$$

   *Hint: relate* $I_4(m^2)$ *to* $(d/dm^2)I_2(m^2)$ *and use the result above.*

   c) Define "renormalized" functions by

   $$\begin{aligned} \widetilde{I}_2(m^2) &= I_2(m^2) + M^2\,\delta_2 + (m^2-M^2)\,\tilde{\delta}_2 \\ \widetilde{I}_4(m^2) &= I_4(m^2) + \delta_4 \ , \end{aligned}$$

   for some constants $\delta_2$, $\tilde{\delta}_2$, $\delta_4$ and the fixed mass parameter $M$. Now choose $\delta_2$, $\tilde{\delta}_2$, and $\delta_4$ such that $\widetilde{I}_2(M^2) = 0$, $\widetilde{I}_4(M^2) = 0$, and $d\widetilde{I}_2/dm^2(M^2) = 0$. These correspond to "renormalization conditions" at the renormalization scale $m^2 = M^2$. With these choices, find the expressions for $\widetilde{I}_2(m^2)$ and $\widetilde{I}_4(m^2)$ at general values of $m^2$ assuming that $\Lambda^2 \gg m^2,\, M^2$. Show that these are finite as $\Lambda \to \infty$, and illustrate what happens to them when $m^2$ becomes much larger than $M^2$

3. Running couplings.

   Consider the $\lambda_0\phi^4$ theory discussed in class, and let us pretend that the exact value of the (1PI connected) 4-point function to one-loop order is

   $$\Gamma^{(4)}(s) = -\lambda + a\lambda^2\left[\ln\left(\frac{\Lambda^2}{s}\right) + C\right] - \delta\lambda \ ,$$

   where $a$ is a positive constant. This is not quite the real answer, but it is close enough for what we want to do here.

   a) Let us choose the renormalization condition to be $\Gamma^{(4)}(s_0) = -\lambda$ for the fixed reference momentum point $s_0$. Solve for $\delta\lambda$ given this condition and show that $\Gamma^{(4)}(s)$ is now finite for any value of $s$ (to one-loop).

b) The renormalization condition above defines the renormalized coupling $\lambda$ in terms of the reference momentum $s_0$. In general, renormalization conditions are not unique – for example we could have chosen a different reference momentum to apply the condition and this would have defined a different renormalized coupling. To account for this, let us define $\lambda(s)$ to be the renormalized coupling defined by the condition $\Gamma^{(4)}(s) = -\lambda(s)$. Derive a relationship between $\lambda(s)$ and $\lambda(s_0)$ by equating $\Gamma^{(4)}(s)$ written in terms of $\lambda(s)$ to the expression you found above for $\Gamma^{(4)}(s)$ written in terms of $\lambda(s_0)$.
*Hint: with this definition the thing we called $\lambda$ above corresponds to $\lambda(s_0)$ here.*

c) Compute $d\lambda/dt$ as a function of $\lambda(s_0)$, where $t = \ln(s/s_0)$. To leading non-trivial order ($\mathcal{O}(\lambda^2)$), we can replace $\lambda(s_0)$ in this expression with $\lambda(s)$. Doing so gives a non-trivial differential equation for $\lambda(s)$. Solve the differential equation for $\lambda(s)$ subject to the boundary condition $\lambda(s = s_0) = \lambda(s_0)$.
*Hint: $\lambda(s_0)$ is a constant with respect to $t$.*

d) Expand your result to $\mathcal{O}(\lambda^2(s_0))$ and show that it reproduces the result from b) at this order. However, note that the expansion parameter in this case is not $\lambda(s_0)$ but rather $\lambda(s_0)\ln(s/s_0)$. What happens to this expansion parameter for $s \gg s_0$?

e) By defining $\lambda(s)$ through the solution to the differential equation rather than directly as in part b), we have extended the range of validity of the perturbative expansion. Even so, perturbativity can still be lost if $\lambda(s)$ grows big. Using the solution from c), at what value of $s$ does $\lambda(s)$ go to infinity? This is called a *Landau pole.*

f) Suppose we had instead that $a = -|a| < 0$. Assuming that $\lambda(s_0)$ is small, what happens to $\lambda(s)$ as $s \to \infty$? What happens as $s$ becomes much smaller than $s_0$?

4. Covariant derivatives and the Higgs. (**Optional - will not be marked!**)

a) For a field tranforming under $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations with generators $t_3^a$, $t_2^p$, and $Y$, write the covariant derivative operator acting on the field in terms of the vector boson mass eigenstates $G_\mu^a$, $W_\mu^\pm$, $Z_\mu^0$, and $A_\mu$. Also, write the $Z_\mu^0$ piece in terms of $\bar{g}$ and $A_\mu$ piece in terms of $e$.

b) Apply the covariant derivative operator to $Q_L$ (assuming one fermion generation) and expand in terms vector boson mass eigenstates as well as the $SU(2)_L$ components $u_L$ and $d_L$.
*Hint: it will be a 2-component vector in $SU(2)_L$ space.*

c) Compute $D_\mu H$ in unitary gauge in terms of the vector boson mass eigenstates. Use this to find $|D_\mu H|^2 \equiv \eta^{\mu\nu}(D_\mu H)^\dagger(D_\nu H)$ in terms of the physical Higgs boson field $h$ and the vector mass eigenstates.

d) Use this result to find the interaction relevant for the decay $h \to W^+W^-$ and compute the corresponding decay width assuming $m_h > 2m_W$. In your answer please write $g$ in terms of $m_W$ and $v$.