PHYS 528 Homework #4

Due: ~~Feb.25, 2021~~ Mar.2, 2021

1. Non-Abelian gauge invariance.

   a) Work out the details and show explicitly that the covariant derivative we discussed for the non-Abelian case transforms according to $D_\mu\psi \to U_r(D_\mu\psi)$.
   Hint: $0 = \partial_\mu(\mathbb{I}) = \partial_\mu(U_r U_r^{-1}) = \partial_\mu(U_r^{-1}U_r)$ *with* $U_r^{-1} = U_r^\dagger$.

   b) We had that $A_{r\,\mu} := A_\mu^a t_r^a \to A'^{\,a}_{r\,\mu} t_r^a = U_r A_{r\,\mu} U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1}$. For $U_r = e^{i\alpha^a t_r^a}$, work out the corresponding transformation of the coefficient functions A_μ^a to linear order in the α^a parameters to derive the result of Eq. (9) in `notes-03` explicitly. Does this result depend on the specific representation chosen? (*i.e.* would the same transformation of the A_μ^a coefficients also work for other representations?)

   c) Fill in the details of the derivation of $[D_\mu, D_\nu]\psi = igt_r^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\, f^{abc} A_\mu^b A_\nu^c)\psi$, for ψ transforming under the rep r of the gauge group.

   d) Write out the covariant derivative acting on a field transforming under the adjoint rep of the non-Abelian group G in terms of the structure constants f^{abc}.

2. Scalar decay to vectors.

   Consider the interaction

$$-\mathscr{L} \supset \frac{1}{\Lambda}\, h\, V_{\mu\nu} V^{\mu\nu} \ , \tag{1}$$

   where h is a real scalar of mass m_h, $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ for some vector boson V_μ, and $\Lambda \gg m_h$ is some very large mass scale. This interaction allows the decay $h(p) \to V_\mu(k_1) + V_\nu(k_2)$, for which the amplitude is

$$-i\mathcal{M} = -\frac{2i}{\Lambda}(k_1^\mu \epsilon_1^\nu - k_1^\nu \epsilon_1^\mu)(k_2^\alpha \epsilon_2^\beta - k_2^\beta \epsilon_2^\alpha)\eta_{\alpha\mu}\eta_{\beta\nu} \ , \tag{2}$$

   where ϵ_1 and ϵ_2 refer to the polarizations of two outgoing vectors.

   a) If V_μ is massless, there are two physical polarization states and a built-in gauge invariance. Compute the summed and squared matrix element "$|\mathcal{M}|^2$" relevant for the total unpolarized decay rate in the h rest frame using the partial completeness relation $\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{\nu\,*}(k,\lambda) = -\eta^{\mu\nu} + (\textit{stuff you can ignore})$, just like what we used for external photons in QED.

   b) A second way to compute the summed and squared matrix element "$|\mathcal{M}|^2$" is to specify external polarization vectors and add up the results. Do this here using the explicit polarization vectors discussed in `notes-04` and summing over all the possibilities.
   Hint: since the initial state is at rest and has no spin, you can choose the $\hat{z}$ *axis to lie along the direction of the first outgoing vector,* $\vec{k}_1 = \|\vec{k}_1\|\,\hat{z}$.

c) Suppose instead that the vector V_μ is massive, with mass m_V. This implies that it has three physical polarization states. The corresponding polarization 4-vectors should satisfy

$$\epsilon(k,\lambda)\cdot\epsilon^*(k,\lambda') = -\delta_{\lambda\lambda'}\ , \qquad k\cdot\epsilon(k,\lambda) = 0\ . \tag{3}$$

For $\vec{k} = \|\vec{k}\|\,\hat{z}$, find a set of three 4-vectors that satisfy these conditions. You should be able to identify two of them as *transverse*, and one as *longitudinal*.

d) Use these three polarization 4-vectors to compute the summed and squared matrix element "$|\mathcal{M}|^2$" for $h \to V_\mu V_\nu$ in the rest frame of the decaying scalar. Also, compare the squared matrix element for longitudinal final states to those for transverse final states.

3. $AA \to \psi\bar{\psi}$ in a general non-Abelian gauge theory with ψ transforming in the rep r.

a) There are two Feynman diagrams for this process: one with the vector in the s-channel and one with the fermion in the t-channel. Find the contribution to the amplitude for $A^a_\mu A^b_\nu \to \psi_i\bar{\psi}_j$ from the s-channel diagram alone.
Hint: the three-point vector interaction is defined for ingoing momenta on all legs. For an outgoing momentum on a leg, just swap $p \to -p$ on that leg.

b) Square this contribution and sum it over all final states and average over initial states (including spin and group), working in the centre-of-mass (CM) frame.
Hint: in the CM frame with vector momenta p_1 and p_2, $(p_1\cdot\epsilon_2) = 0 = (p_2\cdot\epsilon_1)$. Also, $(p_1-p_2)\cdot(p_1+p_2) = 0$ for massless vectors. Use this to simplify the amplitude enormously before squaring.

c) Write down the contribution to the amplitude $A^a_\mu A^b_\nu \to \psi_i\bar{\psi}_j$ from the t-channel diagram alone.

d) Work out the group theory factor corresponding to the t-channel diagram when one squares this contribution and sums/averages it over all final/initial states.