PHYS 528 Homework #3

Due: Feb.11, 2021

1. Consider the theory discussed in ***e.g.* 3.** of `notes-03`:

$$\mathscr{L} = |\partial\phi|^2 - M^2|\phi|^2 + \sum_{i=1}^{2} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - (y\,\phi\bar{\psi}_1\psi_2 + h.c.)\ .$$

   For this theory:

   a) Find the equations of motion for ϕ, ψ_1, and ψ_2.
   Hint: ϕ, ψ_i, ϕ^*, and $\bar{\psi}_i$ should all be treated as independent variables.

   b) Work out the conserved current for the rephasing symmetry discussed in the note.

   c) Show that this current really is conserved.
   Hint: make use of the equations of motion.

   d) Derive the interaction vertices in the theory, and use them to draw the leading-order Feynman diagrams for the processes $\phi + \phi^* \to \psi_1 + \bar{\psi}_1$ and $\phi \to \psi_1 + \bar{\psi}_2$. In both cases, compute the net symmetry charge of the initial and final states.

   e) Starting with the kinetic terms, find the mass dimensions (in natural units) of the scalar and fermion fields. Use this to find the mass dimensions of the mass parameters and couplings in the theory.
   Hint: the action is dimensionless in natural units and $S = \int d^4x\,\mathscr{L}$.

2. Lie algebra stuff.

   a) Show that the structure constants are completely antisymmetric for any Lie group.
   Hint: the first two indices are antisymmetric by their definition. For the last one, multiply the fundamental commutation relation by t^d, take a trace, and keep in mind that traces are cyclic.

   b) Show that for any rep $\{t^a_r\}$, the set of matrices $\{-(t^a_r)^*\}$ also gives a rep.

   c) Prove that the adjoint rep of a compact Lie group is always a real representation.

   d) Prove that the Casimir T^2_r commutes with any generator t^b_r for any rep r.
   Hint: commute through with structure constants, and then use their antisymmetry.

   e) Show that for an irrep of a simple group, $d(r)C_2(r) = d(A)T_2(r)$.
   Hint: contract the Dynkin index equation with δ^{ab}.

3. Connections.

   Consider a fermion in an Abelian gauge theory transforming according to $\psi(x) \to e^{i\alpha(x)}\psi(x) = U(x)\psi(x)$. Suppose there exists an object $P(x,y)$ with $P(x,x) = 1$ and the transformation property $P(x,y) \to U(x)P(x,y)U^{-1}(y)$ for any pair of points x and y. Clearly, $P(x,y)\psi(y) \to U(x)[P(x,y)\psi(y)]$ transforms in the same way as $\psi(x)$ (rather than $\psi(y)$).

a) Define a derivative-like operator $\tilde{D}_\mu$ according to

$$n^\mu \tilde{D}_\mu \psi(x) = \lim_{\epsilon \to 0} \left[P(x, x + \epsilon n)\psi(x + \epsilon n) - \psi(x) \right] / \epsilon$$

where n^μ is any unit 4-vector. The idea behind this definition is that it only really makes sense to take the difference of two objects with the same transformation properties. As far as working out this operator goes, we can expand $P(x, x+\epsilon n) = [1 + i\epsilon n^\mu \tilde{A}_\mu(x)]$ and drop all the higher-order terms in ϵ. Work out $\tilde{D}_\mu$ in terms of ordinary derivatives and the function $\tilde{A}_\mu(x)$. Does this remind you of anything?

b) Under an arbitrary gauge transformation,

$$P(x, x + \epsilon n) \to U(x)P(x, x + \epsilon n)U^{-1}(x + \epsilon n) \equiv [1 + i\epsilon n^\mu \tilde{A}'_\mu(x)].$$

Evaluate $\tilde{A}'_\mu$ in terms of $\tilde{A}_\mu$ and α. Again, does this remind you of anything?