PHYS 528 Homework #1

Due: Jan. 28, 2021

1. Relativistic kinematics.

   a) For the decay $A \to B_1 + B_2$ (with masses M, m_1, and m_2), compute the magnitudes of the outgoing 3-momenta of the decay products as well as their energies in the A rest frame.

   b) Calculate the Mandelstam variables for elastic scattering $\phi(p_1)+\phi(p_2) \to \phi(p_3)+\phi(p_4)$ in the CM frame in terms of the initial 3-momentum magnitude $p = |\vec{p}_1|$ and the scattering angle θ of the outgoing particles relative to the direction of $\vec{p}_1$ (which you can choose to be the z-axis).

2. Consider a theory with two real scalar fields and the Lagrangian

$$\mathscr{L} = \frac{1}{2} Z_{ij} \eta^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_j - \frac{1}{2} M^2_{ij} \phi_i \phi_j \ ,$$

   where $i, j = 1, 2$, and

$$Z = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \ , \qquad M^2 = m^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

   Find the masses of the physical excitations in the theory.

3. Fun with the action.

   a) For the scalar action

$$S[\phi] = \int d^4x \, \left[\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \ ,$$

   integrate the kinetic term by parts so that both derivatives are acting on a single field operator. Also, evaluate the action for the specific field configuration $\phi(x) = a(\vec{k}) \exp(-ik \cdot x)$ for an arbitrary 4-vector $k = (k^0, \vec{k})$ and function $a(\vec{k})$. Show that it vanishes for $k^0 = \pm\sqrt{m^2 + \vec{k}^2}$.

   b) Do all the same things for the massive vector boson action

$$S[A^\mu] = \int d^4x \, \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right] \ ,$$

   with the expansion $A^\mu(x) = \epsilon^\mu(k)\, a(k)\, e^{-ik\cdot x}$.
   Hint: remember the constraint we imposed on massive vectors – what does it translate into for this expansion?

   c) Show that the basic action for a Dirac fermion is real: $S = S^*$ for

$$S[\psi, \overline{\psi}] = \int d^4x \, \overline{\psi} (i\gamma^\mu \partial_\mu - m) \psi \ .$$

   Hint: $(\overline{\psi}_1 \psi_2)^* = (\overline{\psi}_1 \psi_2)^\dagger = (\psi_1^\dagger \gamma^0 \psi_2)^\dagger = \psi_2^\dagger (\gamma^0)^\dagger \psi_1$.

4. Complex scalar theory. (**Optional - will not be marked!**)

$$\mathscr{L} = |\partial\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \ .$$

For this theory:

a) Find the vertex factor for the interaction.
Hint: treat ϕ and ϕ^* as independent variables and keep track of one versus the other with arrows pointing into or out of the vertex, like what is done for fermions.

b) We identify ϕ with a scalar and ϕ^* with its antiparticle, having the same mass but the opposite charge. Feynman rules for a complex scalar are similar to those of a real scalar, but it is necessary to keep track of the difference between ϕ and ϕ^*. To do so, we add arrows to complex scalar lines to track the direction of charge flow. In particular, a ϕ field leads to an arrow pointing into a vertex, and a ϕ^* field gives an arrow pointing out of a vertex. The external leg for a particle in the initial state gets an arrow pointing into the diagram, and a particle in the final state gets an arrow pointing out of the diagram. For antiparticles, the arrow directions are reversed. (This works just like for fermions.) To make an allowed Feynman diagram, the external legs and the legs on vertices must be connected up such that the arrows point in the same direction. Use all this to draw the leading-order Feynman diagrams for $\phi\phi \to \phi\phi$, $\phi^*\phi^* \to \phi^*\phi^*$, and $\phi\phi \to \phi\phi^*$, making sure to draw the arrows correctly, and compute their cross sections.
Hint: compare to the $\lambda\phi^4$ theory – you shouldn't have to compute anything.

d) It might be tempting to try to write a complex scalar field in radial field coordinates,

$$\phi(x) = r(x)\, e^{i\beta(x)/f} \ ,$$

where $r(x)$ and $\beta(x)$ are real scalar fields and f is a constant of mass-dimension one. Show that if you plug this expansion into the standard kinetic term for a complex scalar field, you **do not** obtain canonical (real scalar) kinetic terms for both $r(x)$ and $\beta(x)$.